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SOME INTERESTING PROOFS FROM "YUKTI-BHASHA"

BY

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"Yukti-Bhasha" is the work of an ancient Kerala mathematician, and is conjectured (from internal evidence) to have been written in 1639 A.D. Recently a valuable edition of "Yukti-Bhasha" has been published by the Mangalodayam Press, Trichur. In this paper references are made to this edition. All the theorems, constructions and proofs given here are taken directly from *Yukti-Bhasha* and presented in modern notation with modifications wherever necessary. The proof given for lemma 3 is a modified form of that given in *Yukti-Bhasha.*

1. *Theorem of Pythagoras* (*Ref. Chap. VI, page* 72).—Let ABC be a triangle in which the angle A is a right angle. Suppose AB > AC.

Construction.—On the sides AB and AC describe squares ABDE and ACFG the former on the same side as the hypotenuse and the latter on the side away from it.

Step off GAH = AB, and on AE produced locate K such that EK = AC. Join FH, HD, DK and KF.

Proof.—Now each of the triangles GHF, HBD, DEK and KCF is right-angled, and has one side of length AB and another of length AC.

$\therefore$ All the triangles are congruent to ABC.

$\therefore$ HF = HD = DK = KF = BC.

In the triangle GHF

$$\angle GHF + \angle GFH = 90^\circ.$$

But GFH = BHD

$\therefore$ $\angle$FHD = 90°.

Hence figure FHDK is the square on BC.

Now sq. ABDE + sq. GACF =

$\triangle$ GHF + $\triangle$ HBD + sq. FHDK − $\triangle$ DEK − $\triangle$ KCF

= sq. FHDK

$\therefore$ sq. on AB + sq. on AC = sq. on BC.

2. *Area of a triangle* (*Chap. VII, page* 222).—Let ABC be a triangle in which AD is the perpendicular from A on BC.

It is required to prove that $\triangle$ ABC = $\frac{1}{2}$ BC·AD.

Construction.—Let E′, E be the middle points of BD and AB and F′, F the middle points of CD and AC. Join EE′, FF′. Now we have two triangles BE′E and CF′F.

Rotate triangles BEE′ and CFF′ through 180° about the vertices E and F respectively so that they occupy new positions AEG and AFH where G and H are the new positions of E′ and F′.

Proof.—$\angle$FCF′ = $\angle$FAH

$\angle$E′BE = $\angle$GAE

But $\angle A + \angle B + \angle C = 180°$

$\therefore$ $\angle$A + $\angle$GAE + $\angle$FAH = 180°

$\therefore$ GAH is a straight line parallel to BC,

$\therefore$ Fig. E′H is a rectangle whose area is $\frac{1}{2}$ BC·AD.

But $\triangle$ ABC = rect. E′H

$\therefore$ $\triangle$ ABC = $\frac{1}{2}$ BC·AD.

(*Note:*—In *Yukti-Bhasha* it is assumed that E′H is a rectangle.)

3. *Cyclic Quadrilaterals* (*Ref. Chap. VII, page* 226).

LEMMA 1.—*The difference of the squares on any two chords of a circle = product of the chord of an arc equal to the sum of the two arcs and the chord of an arc equal to the difference of the two arcs.*

Let AC, BC be two chords of a circle meeting at C, AC being the greater chord. Cut off arc AD = arc BC, D being on the same side of AB as C. Join CD. Draw DF, CE ⊥rs to AB from D and E respectively.

Now FE = DC

and AF = EB

$\therefore$ AE = BF.

In the $\triangle$ ACB

$$AE - EB = AE - AF = FE$$
$$= \text{chord of (arc AC} - \text{arc BC)}$$
$$AE + EB = AB \quad = \text{chord of (arc AC} + \text{arc BC)}$$
$$\text{Now } AC^2 - CB^2 \quad = AE^2 - EB^2$$
$$= AB \cdot FE$$
$$= \text{chord of (arc AC} + \text{arc BC)}$$
$$\times \text{chord of (arc AC} - \text{arc BC).}$$

(Note that the proof has to be modified slightly when the chords do not have a common extremity.)

LEMMA 2.—*The product of two chords of a circle is equal to the difference between the squares on the chord of the arithmetic mean of the arcs and on the chord of the semi-difference of the arcs* (Chap. VII, page 228).

Now in the figure of the previous Lemma suppose AB to be one arc and DC to be the other.

$$\text{Then} \qquad \text{sum} = \text{arc AB} + \text{arc DC}$$
$$= \text{arc AC} + \text{arc CB} + \text{arc AC} - \text{arc CB}$$
$$= 2 \text{ arc AC}$$
$$\text{difference} = \text{arc AB} - \text{arc DC}$$
$$= \text{arc AD} + \text{arc CB}$$
$$= 2 \text{ arc CB}$$
$$\therefore \quad \text{arc AC} = \tfrac{1}{2} (\text{arc AB} + \text{arc DC})$$
$$\text{arc CB} = \tfrac{1}{2} (\text{arc AB} - \text{arc DC})$$
$$\therefore \quad AB \cdot DC = (AE + EB)(AE - EB)$$
$$= AE^2 - EB^2 = AC^2 - CB^2$$
$$= \text{sq. on the chord of } \tfrac{1}{2} (\text{arc AB} + \text{arc DC})$$
$$- \text{sq. on the chord of } \tfrac{1}{2} (\text{arc AB} - \text{arc DC}).$$

LEMMA 3.—*In any triangle, the product of two sides divided by circum-diameter = the altitude on the third side* (Ch. VII, p. 231). [Proof—p. 243.]

Let ABC be a triangle with AC as base inscribed in the circle of centre O.

Required to find BM, the perpendicular from B on AC.

Construction.—Draw the diameter EF through the mid-point of BC. Through E draw a parallel to AC to meet the circle at G. Let D be the middle point of arc AC (Fig. 1).

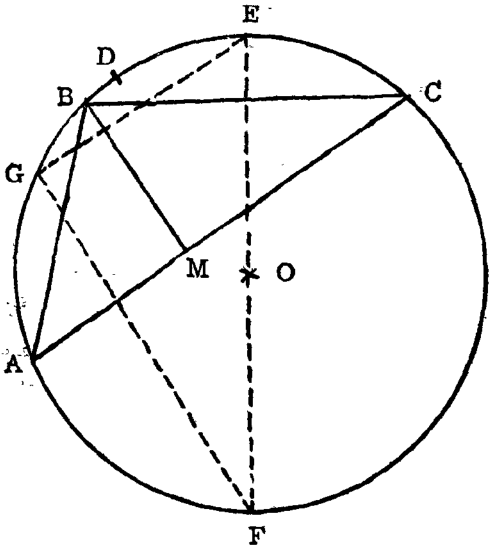

FIG. 1

Proof.—Now arc AD = arc DC

arc AG = arc EC (since EG || to AC)

arc GD = arc AD − arc AG = arc AD − arc EC

= arc AD − $\frac{1}{2}$ arc BC

arc DE = arc DC − arc EC = arc DC − $\frac{1}{2}$ arc BC

arc GD + arc DE = arc AD + arc DC − arc BC

= arc AB

∴ arc GE = arc AB ∴ GE = AB.

The two right-angled triangles EGF, BMC are similar since the corresponding sides are perpendicular.

∴ FE/EG = BC/BM

$$\therefore \quad BM = \frac{EG \times BC}{FE} = \frac{AB \times BC}{FE} = \frac{\text{product of the sides}}{\text{diameter}}$$

4. *The Area of a Cyclic Quadrilateral* (Chap. VII, p. 233).—Let ABCD be a cyclic quadrilateral (Fig. 2). Perform the following constructions.

Mark E and F such that arc AE = arc CD, and arc AF = arc CB. Let H and G be the middle points of arc ED and arc FB.

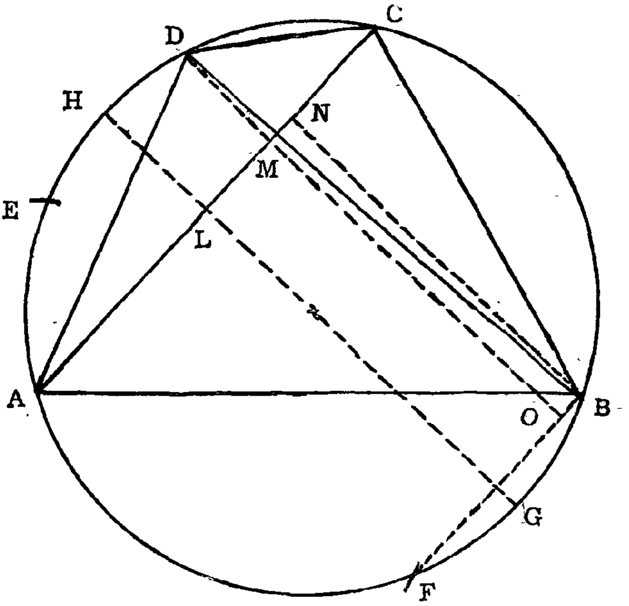

FIG. 2

Then arc HA = arc HC and arc GA = arc GC. Then HG will be a diameter and it is the perpendicular bisector of both AC and FB.

Let HG meet AC at L. Draw DO $\perp$ to AC and FB meeting AC at M and FB at O. From B draw BN $\perp$ to AC.

Proof.—Now

the quadrilateral ABCD $= \triangle ADC + \triangle ABC$

$$= \tfrac{1}{2} AC \cdot DM + \tfrac{1}{2} AC \cdot BN;$$

since AC and FB are parallel (because arc AF = arc CB), BN = OM.

$$\therefore \quad ABCD = \tfrac{1}{2} AC \cdot DM + \tfrac{1}{2} AC \cdot OM$$

$$= \tfrac{1}{2} AC \cdot DO$$

By lemma 3, DO = (DF · DB)/diameter

$$\therefore \quad ABCD = \frac{AC \cdot DF \cdot DB}{2 \text{ diameter}}$$

Let FD be called the *third diagonal*

$$\therefore \text{Area} = \frac{\text{Product of the three diagonals}}{2 \text{ diameter}}$$

To find the lengths of the diagonals

$$AD \cdot DC = AH^2 - EH^2 \text{ (by lemma 2)}$$

$$\left[\because \begin{array}{l} AH = \text{chord of } \tfrac{1}{2} (\text{arc } AD + \text{arc } DC) \\ EH = \text{chord of } \tfrac{1}{2} (\text{arc } AD - \text{arc } DC) \end{array} \right]$$

Similarly

$$AB \cdot BC = AG^2 - GB^2$$

$$\therefore \; AD \cdot DC + AB \cdot BC = AG^2 + AH^2 - EH^2 - GB^2$$

But $AG^2 + AH^2 = HG^2 = \text{diameter}^2 = d^2$

$$\therefore \; AD \cdot DC + AB \cdot BC = d^2 - EH^2 - GB^2$$

Now $d^2 - EH^2 = d^2 - HD^2 = GD^2$

$$\therefore \; AD \cdot DC + AB \cdot BC = GD^2 - GB^2 = \text{chord of } (\text{arc } GD + \text{arc } GB) \times \text{chord of } (\text{arc } GD - \text{arc } GB)$$

$$= \text{chord of } (\text{arc } GD + \text{arc } FG) \times \text{chord of } (\text{arc } GD - \text{arc } GB)$$

$$= FD \cdot BD.$$

But FD $= \text{chord of } (\text{arc } AF + \text{arc } AD)$

$= \text{chord of } (\text{arc } CB + \text{arc } CE)$

$= BE$

FD and BE may both be called the *third diagonal.*

There cannot be a *fourth diagonal* by transposing the chords in any other way.

Let BD be d_1, AC be d_2 and FD be d_3.

$$AD \cdot DC + AB \cdot BC = d_1 \cdot d_3 \tag{1}$$

Similarly $AD \cdot AB + CB \cdot CD = d_2 \cdot d_3$ (2)

Again in the quadrilateral ADCF,

$$CF = AB, \; AF = CB.$$

But $CF \cdot CD + AF \cdot AD = AC \cdot BD.$

$$\therefore \; AB \cdot CD + BC \cdot AD = AC \cdot BD = d_1 \cdot d_2 \tag{3}$$

Let the sides AB, BC, CD, AD be denoted by a, b, c, d,

Then $$cd + ab = d_1 d_3 \qquad (1')$$

$$da + bc = d_2 d_3 \qquad (2')$$

$$ac + bd = d_1 d_2 \qquad (3')$$

$(1') \times (3')$ gives

$$d_1^2 d_2 d_3 = (ac + bd)(ab + cd)$$

$$\therefore \quad d_1^2 = \frac{(ac + bd)(ab + cd)}{(ad + bc)} \qquad (4)$$

$$\therefore \quad d_1 = \sqrt{\frac{(ac + bd)(ab + cd)}{(ad + bc)}}$$

Similarly we can find d_2 and d_3.

GRASSMANN CUBIC AND WALLACE LINES*

BY

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INTRODUCTION

1. It is well known that the feet of the perpendiculars from any point P on the circumcircle of a triangle ABC to its sides lie on a line called the Wallace line of P.[1] Also the locus of a point P the feet of the perpendiculars from which to the sides of a triangle ABC are collinear is its circumcircle. The same is true if, instead of perpendiculars, obliques inclined at the same angle θ in the same sense are drawn, the line of collinearity in this case being called the Poncelot line of P[2,3] or the θ-pedal line of P. In any case the envelope of the line of collinearity is a curve of the third class with the line at infinity as a bitangent, the points of contact being the circular points. In other words the envelope is a three cusped hypocycloid.

Many generalisations of this theorem and its converse are known. By regarding the circular points as any two ordinary points a purely projective generalisation for a circumconic can be derived.[4] Takasu[5] has shown that Aubert's theorem[6]: "If O be a fixed point in the plane of a conic and lines OAA′, OBB′, OCC′ meet the conic in AA′, BB′, CC′, and if P be any point on the conic, then LMN, the points of intersection of PA′ with BC, PB′ with CA, PC′ with AB lie on a line through O", is a projective generalisation of the Wallace line theorem and obtains various projective and affine generalisations of the theorem. Ono proves by using trilinear co-ordinates the result that the locus of a point P lines drawn through which parallel to QA, QB, QC where Q is a fixed point in the plane of the triangle ABC meet the sides BC, CA, AB in collinear points is a circumconic.[7] Using Morley's co-ordinates Ogino obtains the theorem that if instead of parallels, lines inclined at an angle θ in the same sense with QA, QB, QC meeting the sides of the triangle in collinear points be drawn through a variable point P, its locus is also a circumconic.[8]

A more comprehensive generalisation of the theory of the pedal line is the one suggested by Grassmann's method of generation of the cubic, *viz.*, If ABC, A′B′C′ be two triangles in the same plane the locus of a point P which moves such that LMN, the intersections respectively of PA′, BC;

* I take this opportunity to thank Dr. R. Vaidyanathaswamy for his guidance and suggestions in the preparation of this paper and the authorities of the Madras University for awarding me a fellowship during which this was one of the investigations carried out

PB′, CA; PC′, AB are in a line is a cubic curve *circumscribing both the* triangles. Berkhan,[9] Sommerville[10] and Gabbat,[11] the first two analytically and the last by synthetic methods have developed the theory of the pedal line in the non-euclidean plane by considering the two triangles in Grassmann's method to be two triangles in perspective. The pedal line theory in this case is symmetric with respect to the two triangles.

We shall however proceed directly to obtain the different generalisations in the euclidean plane as well as the main theorem itself by regarding one of the two triangles in Grassmann's method as a line triangle. We conclude with a demonstration that Aubert's theorem is a particular case of Grassmann's method of generation of the cubic when the two triangles are at once inscribed in a conic and in perspective. In this case both the Grassmann cubic locus and envelope degenerate—the one into a line and a conic locus, the other into a point and a conic envelope.

2. GRASSMANN CUBIC LOCUS OF TWO TRIANGLES

If ABC, A′B′C′ are two triangles in the same plane, the locus of a point P such that PA′, PB′, PC′ meet BC, CA, AB respectively in collinear points LMN is a cubic curve. We shall call it the Grassmann cubic locus of the two triangles and the line of collinearity the generalised Wallace line of P w.r.t. the two triangles. By inspection we can obtain nine points on the cubic, *viz.*, ABC, A′B′C′, A″B″C″ where A″B″C″ are respectively the intersections of BC, B′C′; CA, C′A′; AB, A′B′. Also these nine points determine a unique cubic; a particular cubic, *viz.*, the three lines A″BC, A′B″C′, and A′B′C″ containing eight of the nine points does not contain the ninth point A.

Let ABC be taken as the triangle of reference and A′, B′, C′ be $x_r y_r z_r$ $(r = 1, 2, 3)$. Let P be $x'y'z'$. The lines PA′, PB′, PC′ *are given respectively* by $\Sigma x\,(y'z_r - z'y_r) = 0$ $(r = 1, 2, 3)$. If LMN are respectively the intersections of PA′ with BC, PB′ with CA, PC′ with AB, they are given by

$$\frac{x}{0} = \frac{y}{x'y_1 - y'x_1} = \frac{z}{x'z_1 - z'x_1}$$

$$\frac{x}{y'x_2 - x'y_2} = \frac{y}{0} = \frac{z}{y'z_2 - z'y_2}$$

$$\frac{x}{z'x_3 - x'z_3} = \frac{y}{z'y_3 - y'z_3} = \frac{z}{0}$$

Let $lx + my + nz = 0$ be the line of collinearity of LMN. Then

$$\left.\begin{aligned} m\,(x'y_1 - y'x_1) + n\,(x'z_1 - z'x_1) &= 0 \\ n\,(y'z_2 - z'y_2) + l\,(y'x_2 - x'y_2) &= 0 \\ l\,(z'x_3 - x'z_3) + m\,(z'y_3 - y'z_3) &= 0 \end{aligned}\right\} \qquad 2{\cdot}1$$

Eliminating $l\,m\,n$ from 2·1 and dropping dashes the locus of P is found to be

$$\begin{vmatrix} 0 & xy_1 - x_1y & xz_1 - x_1z \\ yx_2 - y_2x & 0 & yz_2 - y_2z \\ zx_3 - z_3x & zy_3 - z_3y & 0 \end{vmatrix} = 0 \qquad 2\cdot 2$$

Expanding, we have

$$(xy_1 - x_1y)(yz_2 - y_2z)(zx_3 - z_3x) + (xz_1 - x_1z)(yx_2 - y_2x)(zy_3 - z_3y) = 0. \qquad 2\cdot 3$$

This equation 2·3 shows that the cubic passes through the nine intersections of CA′, AB′, BC′ with BA′, CB′, AC′, *viz.*, ABC A′B′C′ XYZ where XYZ are respectively the points of intersection of BC′, B′C; CA′, C′A; AB′, A′B. Thus three more points XYZ on the cubic are known.

3. Clebsch[12] has proved the converse of Grassmann's theorem: Given a cubic and a triangle inscribed in it, it is possible to find three other triangles which are such that, with any one of them and the given triangle the cubic can be described according to Grassmann. For in Grassmann's construction if BC, CA, AB meet the cubic again in A″B″C″ the required triangles are those whose vertices lie on the cubic and whose sides pass through these three points. Using elliptic parameters he has shown that there are four triangles whose sides pass through three given points. Thus if the given triangle is one of them there are three other triangles satisfying the given conditions. If $\alpha\beta\gamma$ are the elliptic parameters of the vertices of the given triangle, those of the other triangles are got by the addition of half periods.

If however A″B″C″ are in a line[13] there are only three proper triangles, the fourth coinciding with the line △ A″B″C″.

4. GRASSMANN'S CUBIC ENVELOPE OF TWO TRIANGLES

As P moves on the Grassmann cubic locus of the two triangles, the line of collinearity envelopes a curve of the third class. By eliminating x', y', z' from 2·1 we obtain the equation of the envelope to be

$$\begin{vmatrix} -\left(\dfrac{my_1 + nz_1}{x_1}\right) & m & n \\ l & -\left(\dfrac{nz_2 + lx_2}{y_2}\right) & n \\ l & m & -\left(\dfrac{lx_3 + my_3}{z_3}\right) \end{vmatrix} = 0. \qquad 3\cdot 1$$

We shall call it the Grassmann cubic envelope of the two △s. It is easy to verify that this cubic envelope touches the six sides of the two triangles and in addition the lines AA′, BB′, CC′. Also these nine lines determine the cubic uniquely in general; for one cubic envelope consisting of the three

points $AB'C'$ lies on the eight lines $B'C'$, $C'A'$, $A'B'$, AB, AC, AA', BB', CC' and does not lie on the ninth line BC.

Now we attempt to solve the converse question: Given a cubic envelope and three tangents AB, BC, CA to it, in how many ways is it possible to find another triangle such that the given cubic is the Grassmann cubic envelope of the two triangles. We have seen that if ABC, $A'B'C'$ are the two triangles the cubic envelope in question touches AA', BB', CC' in addition to the sides of the two triangles themselves. Hence if we draw the remaining three tangents one from each of the vertices of the given triangle, the required triangle must have its vertices on these lines and its sides must touch the cubic. This is precisely the problem reciprocal to that solved by Clebsch and hence we have in this case, as in the one above, three other triangles each of whose Grassmann cubic envelope with the given triangle is the given cubic.

If however the three tangents one from each of the vertices of the given triangle to the cubic envelope are concurrent, we have only two other triangles.

5. A particular case of importance for us is the one in which $A'B'C'$ are in a line. Of the nine points ABC, $A'B'C'$, $A''B''C''$ which determine the Grassmann cubic locus, the last six of them in this case lie on the line of collinearity $A'B'C'$. Hence the cubic locus breaks up into the given line and a conic circumscribing ABC.

Let $px + qy + rz = 0$ be the line $A'B'C'$. Then $px_s + qy_s + rz_s = 0$ $(s = 1, 2, 3)$. Multiplying the first column of $2\cdot2$ by p, the second by q and the third by r, adding and removing the common factor $px + qy + rz$, we find that $2\cdot2$ becomes

$$(px + qy + rz)\begin{vmatrix} x_1 & xy_1 - x_1y & xz_1 - x_1z \\ y_1 & 0 & yz_2 - y_2z \\ z_3 & zy_3 - z_3y & 0 \end{vmatrix} = 0.$$

The determinant factor denotes evidently a conic circumscribing ABC. Thus we have the theorem:

When one of the two triangles in Grassmann's method of generation of the cubic is a collinear triad, the cubic locus breaks up into this line of collinearity and a conic circumscribing the triangle ABC.[14]

We shall call it the Grassmann conic of the given triangle and the collinear triad.

6. We shall prove that the three points forming the line Δ and the intersections of the line of collinearity with the sides of the given triangle determine a homography of order 3, with the intersections of the line with the Grassmann conic as double points. We know that XYZ the inter-

sections of BC′, B′C; CA′, C′A; AB′, A′B lie on the conic. Consider the homographic ranges determined by ABC, XYZ as corresponding points. Their homographic axis is the line of collinearity A′B′C′. Let $\alpha\beta$ be the intersections of this line with the conic and $A_1B_1C_1$ its intersections with BC, CA, AB. Project the line A′B′C′ into the line at infinity and the conic into a circle and represent their projections by small letters. Then ay, cx, bz are respectively parallel to bx, az, cy so that $\angle a'ba_1 = \angle b'cb_1 = \angle c'ac_1$ and hence $(A'\alpha A_1\beta) = (B'\alpha B_1\beta) = (C'\alpha C_1\beta) = k$ say. Thus $\alpha\beta$ are the double points of the homography determined by $A_1B_1C_1$ and A′B′C′ as corresponding points. This homography becomes an involution if AA′, BB′, CC′ are concurrent. For by the same projection we have the three lines aa', bb', cc' inclined at the same angle with the sides bc, ca, ab. Making use of the fact that they are concurrent we see that this angle must be $\pi/2$ and hence $k = -1$, the homography thus reducing to an involution.

Also when the homography becomes an involution, aa', bb', cc' in the projected figure are perpendicular to the opposite sides and are hence concurrent. We are thus led to the theorem:

The homography on the given line which carries the intersections of the sides of the given triangle into the collinear triad has the intersections of the Grassmann conic as double points. This homography becomes an involution if and only if AA′, BB′, CC′ are concurrent.

7. We proceed to determine the Grassmann cubic envelope in this case. When A′B′C′ lie on the line $px + qy + rz = 0$, 3·1 is satisfied by the values pqr of lmn and hence the line of collinearity is a tangent to this envelope.

Writing the first row of 3·1 as it is, subtracting the first row from the second and writing it as the second row, subtracting the second row from the third and writing it as the third row we have

$$\begin{vmatrix} -\left(\dfrac{my_1+nz_1}{x_1}\right) & m & n \\ \dfrac{\Sigma lx_1}{x_1} & -\dfrac{\Sigma lx_2}{y_2} & 0 \\ 0 & \dfrac{\Sigma lx_2}{y_2} & -\dfrac{\Sigma lx_3}{z_3} \end{vmatrix} = 0.$$

Expanding and rearranging, we have

$$F(l, m, n) \equiv my_2\Sigma lx_3\,\Sigma lx_1 - my_1\Sigma lx_3\,\Sigma lx_2 + nz_3\,\Sigma lx_2\,\Sigma lx_1 - nz_1\,\Sigma lx_2\,\Sigma lx_3 = 0 \qquad 7{\cdot}1$$

$$\frac{\partial F}{\partial l} = my_2\,(x_3\Sigma lx_1 + x_1\,\Sigma lx_3) - my_1\,(x_3\,\Sigma lx_2 + x_2\,\Sigma lx_3) + nz_3\,(x_2\,\Sigma lx_1 + x_1\,\Sigma lx_2) - nz_1\,(x_2\,\Sigma lx_3 + x_3\,\Sigma lx_2).$$

This vanishes for the values pqr of lmn. Expressions for $\frac{\partial F}{\partial m}, \frac{\partial F}{\partial n}$ are also found to vanish for the same values of lmn. Hence the given line $px + qy + rz = 0$ is a bitangent of the cubic envelope.

We now prove that $\alpha\beta$ are the points of contact of the bitangent with the cubic envelope. We know that this envelope touches BC, CA, AB and AA′, BB′, CC′ with A′B′C′ as a bitangent. Also the double points of the homographic ranges $(A_1B_1C_1 \ldots .)$, $(A'B'C' \ldots .)$ are $\alpha\beta$. Consider the reciprocal problem: a cubic has a node at a given point N and circumscribes a triangle PQR. The sides of the triangle meet the cubic again in P′Q′R′. We are required to prove that the nodal tangents are the double lines of the homographic pencils N (PQR. . . .), N (P′Q′R′. . . .).

Consider the double lines of the three involutions N (QR′. . . .), N (Q′R. . . .); N (RP′. . . .), N (R′P. . . .) and N (PQ′. . . .), N (P′Q. . . .). It is a known theorem that the double lines of these involution pencils themselves form an involution whose double lines are the double lines of the original homographic pencils. We know that every triad of collinear points on a nodal cubic is apolar to the three inflexions. The apolar complements therefore of P are QR′ and Q′R. If XX′ be the first polar of P, they are also apolar to QR′ and Q′R. Similarly the first polars of Q and R, *viz.*, YY′ and ZZ′ are apolar to RP′, R′P and PQ′, P′Q respectively. But the Hessian pair of the inflexional triad is apolar to the first polar of every point. Hence the Hessian pair are the double points of the involution of which XX′, YY′, ZZ′ are point pairs. Hence the nodal tangents are the double lines of the original homographic pencil. Thus we are led to the theorem:

> *When one of the triangles in Grassmann's method becomes a line triangle, the cubic envelope has the given line as a bitangent, the points of contact being the intersections of the given line with the Grassmann conic.*

8. Let us now try to find how many triads of points *on a given line* can be found such that the degenerate cubic consisting of the given line and a conic can be described according to Grassmann. We have proved in §6 that if the line of collinearity meets BC, CA, AB in $A_1B_1C_1$ and the conic in $\alpha\beta$ then $(A'\alpha A_1\beta) = (B'\alpha B_1\beta) = (C'\alpha C_1\beta)$. We are therefore required to find A′B′C′ satisfying the above conditions. Take any point A′ at random on the given line. Join BA′, CA′ meeting the conic again in YZ respectively. Join AY, AZ to meet the given line in B′C′. Then A′B′C′ are the required points.

Proof.—Let BC′ meet the conic again in X. In the first place we prove that B′C also meets the conic in X. In other words B′CX are collinear. Consider the hexagon BYAZCX. Now BY, CZ meet at A′; AZ, XB meet at C′. Let AY, CX meet at B″. Then A′C′B″ are collinear i.e. B″

lies on $A'C'$. It lies on AY also. Therefore B'' coincides with B'. Hence $B'CX$ are collinear. We can then prove in the same way as in §6 that $(A'\alpha A_1\beta) = (B'\alpha B_1\beta) = (C'\alpha C_1\beta)$.

But since A' can be chosen in ∞^1 ways, and when once A' is chosen the remaining points $B'C'$ are uniquely determined, the triads themselves can be chosen in ∞^1 ways. In fact so long as $A'B'C'$ move on the given line such that together with $A_1B_1C_1$ they determine homographic ranges with $\alpha\beta$ as double points, they can be taken as a collinear triad of the given conic considered as the Grassmann conic. Hence the theorem:

Given a line and a conic with a triangle inscribed in it, ∞^1 triads of points on the given line can be found, which together with the given triangle generate the cubic consisting of the line and the conic according to Grassmann.

9. We are now in a position to describe the most general projective theory of the pedal lines for any conic. Consider a line and a conic with the triangle ABC inscribed in it. Let the sides of the triangle meet the line in $A_1B_1C_1$. Let the intersections of the line with the conic be $\alpha\beta$. Choose points $A'B'C'$ such that

$$(A'\alpha A_1\beta) = (B'\alpha B_1\beta) = (C'\alpha C_1\beta) = k \text{ say.}$$

Then any point P on the conic (or the given line) is such that PA', PB', PC' meet BC, CA, AB in three collinear points. We shall call the line of collinearity the generalised k-Wallace line of P. The envelope of these lines is a tricuspidal quartic having the given line as a bitangent with the points of contact at $\alpha\beta$ since the only third class curve with a bitangent is a tricuspidal quartic. As the collinear triad $A'B'C'$ varies, we get ∞^1 tricuspidal quartics all inscribed in the given triangle and having the same points of contact $\alpha\beta$ with the bitangent $A_1B_1C_1$.

An affine generalisation of the pedal line theory is obtained by regarding the given line as the line at infinity. Let ABC be a triangle inscribed in a conic meeting the line at infinity in $A_1B_1C_1$. Let $A'B'C'$ be points such that $(A'\alpha A_1\beta) = (B'\alpha B_1\beta) = (C'\alpha C_1\beta)$. Then any point P on the conic as well as on the line at infinity is such that PA', PB', PC' meet BC, CA, AB in three collinear points. We shall call this the pedal line of P. We regard the conic as the orthogonal projection of a circle and define the direction of a line as given by the eccentric angle of either extremity of the parallel diameter. With this definition the direction of a line or what is the same the direction of the point at infinity is defined modulo π. Let the projections of A_1A', B_1B', C_1C' be respectively a_1a', b_1b', c_1c'. Let $\alpha\beta$ be the projections of the circular points ij. Then $(A_1A', \alpha\beta) = (a_1a', ij) = e^{2i(\theta-\theta_1)}$, where θ' and θ_1 are the directions of A' and A_1. Hence the relation $(A_1A', \alpha\beta)$

$=(B_1B', \alpha\beta) = (C_1C', \alpha\beta)$ can now be interpreted as $\theta' - \theta_1 = \phi' - \phi_1 = = \psi' - \psi_1 = \lambda$ say, where each of these angles is determined modulus π. $\therefore\ \theta' = \theta_1 + \lambda,\ \phi' = \phi_1 + \lambda,\ \psi' = \psi_1 + \lambda$ say. Hence the directions of the triads A′B′C′ are obtained from those of the sides of the triangle ABC by the addition of the same angle λ. As λ varies we obtain all such triads of points. We define the line of collinearity as the λ-pedal line of P.

The envelope of such lines is a tricuspidal quartic with the line at infinity as a bitangent, the points of contact being the intersections $\alpha\beta$ of the conic with the line at infinity.

Given such a tricuspidal quartic with its points of contact with the line at infinity (which is a bitangent) and any three tangents to it, it is possible to regard it as the θ-pedal envelope of a unique conic. For if ABC be the triangle formed by the three given tangents, draw the third tangent that can be drawn from each vertex to meet the line at infinity in A′B′C′. Also let the sides of the triangle meet the line at infinity in $A_1B_1C_1$. Then by §7, we have $(A'A, \alpha\beta) = (B'B, \alpha\beta) = (C'C, \alpha\beta) = e^{2i\theta}$ say. We have then for the unique conic through $ABC\alpha\beta$ the given tricuspidal quartic as the θ-pedal envelope.

It is now easy to obtain the Poncelot line of P in the case of the circum-circle of a triangle ABC. Let A′B′C′ be chosen on the line at infinity such that $(A'A_1, ij) = (B'B_1, ij) = (C'C_1, ij)$ i.e. A′B′C′ are at the same angular distance from the sides of the triangle ABC. The locus of P such that PA′, PB′, PC′ meet BC, CA, AB in three collinear points is by §6 a circum-conic through ij i.e. the circumcircle of ABC. Also by §8, if P be any point on the circumcircle of ABC, lines PA′, PB′, PC′ i.e. lines drawn through P inclined at the same angle θ in the same sense meet BC, CA, AB in a line called the Poncelot line of P. By §7 the envelope of such lines is a curve of the third class with the line at infinity as a bitangent, the points of contact being the circular points. In other words, the Grassmann cubic envelope is a three-cusped hypocycloid. If however the angle θ is $\pi/2$, the line of collinearity is the Wallace line of P and the Grassmann cubic envelope is the well-known three-cusped hypocycloid touching the sides of the triangle ABC and AA′, BB′, CC′ which are now the altitudes of the triangle.

We are now in a position to exhibit the theorems of Ono and Ogino as particular cases of our general result.

ONO'S THEOREM.—*Given a point Q in the plane of the* $\triangle ABC$, *draw through P parallels PL, PM, PN to QA, QB, QC cutting the sides* (i) *BC, CA, AB in points* $L_1M_1N_1$, (ii) *CA, AB, BC in* $M_2N_2L_2$ *and* (iii) *AB, BC, CA in points* $N_3L_3M_3$. *Then the locus of P such that* $L_1M_1N_1$, $L_2M_2N_2$ *and* $L_3M_3N_3$ *are collinear is in each case a circum-conic of ABC.*

Let QA, QB, QC meet the line at infinity in A′B′C′ respectively. Find the double points $\alpha_1\beta_1$; $\alpha_2\beta_2$; $\alpha_3\beta_3$ of the homographic ranges

$$(A_1B_1C_1\ldots.) = (A'B'C'\ldots.)$$
$$(A_1B_1C_1\ldots.) = (C'A'B'\ldots.)$$
$$(A_1B_1C_1\ldots.) = (B'C'A'\ldots.)$$

where $A_1B_1C_1$ are the points at infinity on BC, CA, AB. Then by §5 and §6 the loci of P when $L_1M_1N_1$; $L_2M_2N_2$; $L_3M_3N_3$ are collinear are respectively the three conics circumscribing the triangle ABC and passing through $\alpha_1\beta_1$; $\alpha_2\beta_2$; $\alpha_3\beta_3$. In each case the points at infinity which correspond are by §6 in the directions of conjugate diameters of the respective conics.

Ogino's extension.—Consider points $A_0B_0C_0$ on the line at infinity at an angular distance θ_0 from A′B′C′. Then $A_0B_0C_0$ are fixed points at infinity. By §5 the locus of a point P such that lines drawn through it inclined at an angle θ_0 in the same sense to QA, QB, QC meet BC, CA, AB in collinear points is a circum-conic of ABC. In fact Ogino's extension can further be generalised. Since all that is necessary is to fix three points at infinity, if lines be drawn through a variable point P inclined at angles θ, ϕ, ψ with QA, QB, QC to meet the sides in collinear points, the locus of P is a circum-conic of ABC. If however the conic is given, the method of determination of A′B′C′ is dealt with in §9.

We shall now indicate how Aubert's theorem can be obtained by regarding the two triangles in Grassmann's method of generation of the cubic to be two triangles inscribed in a conic and in perspective. Consider any conic and three secants drawn from a fixed point O to meet the conic in AA′, BB′, CC′. Evidently the two triangles ABC, A′B′C′ are in perspective. Let us obtain the Grassmann cubic locus of these two triangles. It passes through ABC, A′B′C′, A″B″C″ and XYZ. The six points A″B″C″, XYZ lie on the polar of O for the conic by the quadrangle construction for pole and polar. Hence the Grassmann cubic locus breaks up into the polar of O (which is the axis of perspective of the two triangles) and the given conic.

The Grassmann cubic envelope also degenerates in this case. For it must touch the six sides of the two triangles ABC and A′B′C′ and the lines AA′, BB′, CC′. The line of collinearity when P assumes the position A′ is the line joining the intersections of A′B′ with AC say X, and A′C′ with AB say X′, and similarly for B′ and C′. Let XX′ meet AA′ at O′ and let AB, A′B′ meet at L; AC, A′C′ meet at M. Then LM is the polar of O. Let AA′ meet LM at X_1. Then by the harmonic property of the quadrangle ALA′M we have $(X_1A'O'A) = -1$. But $(X_1A'OA)$ is also -1. Hence O and O′ coincide or XX′ passes through O. Similarly the lines of collinearity as P assumes the positions B′ and C′ also pass through O. Hence the

cubic envelope must degenerate into O and the conic inscribed in the two triangles.

But we have seen that the lines of collinearity corresponding to points ABC, A′B′C′ pass through O and those corresponding to points on the axis of perspective say A″B″C″ touch the conic. Hence by the principle of continuity, lines of collinearity corresponding to points on the conic will always pass through the fixed point O. Hence Aubert's theorem.

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ON EQUAL SUMS OF LIKE POWERS

BY

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1. We write

$(m)^k = (n)^k$ when there exist positive integers $x_s\,(s \leqslant m)$, $y_t\,(t \leqslant n)$ (1)

such that

$$\sum_{s \leqslant m} x_s^k = \sum_{t \leqslant n} y_t^k \text{ where } (x_1, \ldots, x_m, y_1, \ldots, y_n) = 1$$

$$x_s\,(s \leqslant m) \neq y_t\,(t \leqslant n).$$

If (1) has infinitely many solutions we write

$$(m)^k = (n)^k \; i.o. \tag{2}$$

Further if the equations

$$a_1^h + \ldots + a_j^h = b_1^h + \ldots + b_j^h \qquad (1 \leqslant h \leqslant k) \tag{3}$$

have a non-trivial solution in positive integers, *i.e.*, a solution in which the b's are not a permutation of the a's we write

$$[a_1, a_2, \ldots, a_j]_k = [b_1, b_2, \ldots, b_j]_k. \tag{4}$$

We prove the following theorem.

THEOREM 1.—$(6)^7 = (6)^7$ *i.o.*

I. Chowla* has proved a similar result for sixth powers namely $(5)^6 = (5)^6$ *i.o.*

2. A solution of (4) for $k = 7$ is given by T. L. Burchnall and T. W. Chaundy† in one parameter as follows:—

$$\begin{aligned}
&[-15\lambda^2 + 28\lambda - 21,\ -12\lambda^2 + 18\lambda + 14,\ -9\lambda^2 + 32\lambda - 7,\\
&\quad -6\lambda^2 - 10\lambda + 28,\ 9\lambda^2 - 32\lambda + 7,\ 12\lambda^2 - 18\lambda - 14,\\
&\quad 6\lambda^2 + 10\lambda - 28,\ 15\lambda^2 - 28\lambda + 21]_7\\
&\quad = [-15\lambda^2 + 24\lambda + 7,\ -12\lambda^2 + 38\lambda - 14,\ -9\lambda^2 - 4\lambda + 21,\\
&\qquad -6\lambda^2 + 14\lambda - 28,\ 6\lambda^2 - 14\lambda + 28,\ 9\lambda^2 + 4\lambda - 21,\\
&\qquad 12\lambda^2 - 38\lambda + 14,\ 15\lambda^2 - 24\lambda - 7]_7.
\end{aligned} \tag{5}$$

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* I. Chowla, *On sums of sixth powers*, J. L. M. S., **10**, 233.

† *The Quarterly Journal of Mathematics*, **8**, 119–30.

Tarry has observed that (5) implies

$$[a_1+z,\ldots,a_j+z]_k=[b_1+z,\ldots,b_j+z]_k. \tag{6}$$

$$[a_1,\ldots,a_j,\ b_1+d,\ldots,b_j+d]_{k+1}=[b_1,\ldots,b_j,\ a_1+d,\ldots,a_j+d]_{k+1}. \tag{7}$$

From (6) and (7) with $z=15\lambda^2+38\lambda+28$ we get

$$\begin{aligned}&[66\lambda+7,\ 3\lambda^2+56\lambda+42,\ 6\lambda^2+70\lambda+21,\ 9\lambda^2+28\lambda+56,\\ &\quad 21\lambda^2+48\lambda,\ 24\lambda^2+6\lambda+35,\ 27\lambda^2+20\lambda+14,\\ &\quad 30\lambda^2+10\lambda+49]_7\\ &=[62\lambda+35,\ 3\lambda^2+76\lambda+14,\ 6\lambda^2+34\lambda+49,\\ &\quad 9\lambda^2+52\lambda,\ 21\lambda^2+24\lambda+56,\ 24\lambda^2+42\lambda+7,\ 27\lambda^2+42,\\ &\quad 30\lambda^2+14\lambda+21]_7.\end{aligned} \tag{8}$$

Now from (7) and (8) with $d=3\lambda^2+14\lambda-21$, we have

$$\begin{aligned}&[62\lambda+35,\ 3\lambda^2+80\lambda-14,\ 6\lambda^2+34\lambda+49,\ 9\lambda^2+52\lambda,\\ &\quad 9\lambda^2+84\lambda,\ 12\lambda^2+42\lambda+35,\ 21\lambda^2+24\lambda+56,\ 24\lambda^2+42\lambda+7,\\ &\quad 24\lambda^2+62\lambda-21,\ 27\lambda^2+42,\ 30\lambda^2+34\lambda-7,\ 33\lambda^2+24\lambda+28]_8\\ &=[66\lambda+7,\ 3\lambda^2+56\lambda+42,\ 6\lambda^2+90\lambda-7,\ 9\lambda^2+28\lambda+56,\\ &\quad 9\lambda^2+48\lambda+28,\ 12\lambda^2+66\lambda-21,\ 21\lambda^2+48\lambda,\ 24\lambda^2+6\lambda+35,\\ &\quad 24\lambda^2+38\lambda+35,\ 27\lambda^2+56\lambda-14,\ 30\lambda^2+10\lambda+49,\\ &\quad 33\lambda^2+28\lambda]_8\end{aligned} \tag{9}$$

which is a solution of (4) with $k=8$, $j=12$.

Then applying (6) to (9) with $z=-\dfrac{33\lambda^2+90\lambda+35}{2}$

we get

$$\begin{aligned}&(33\lambda^2-34\lambda-35)^k+(27\lambda^2-70\lambda+63)^k+(21\lambda^2+22\lambda-63)^k\\ &\quad+(15\lambda^2-14\lambda+35)^k+(15\lambda^2-78\lambda+35)^k+(9\lambda^2+6\lambda-35)^k\\ &=(33\lambda^2-42\lambda+21)^k+(27\lambda^2-22\lambda-49)^k+(21\lambda^2-90\lambda+49)^k\\ &\quad+(15\lambda^2+34\lambda-77)^k+(15\lambda^2-6\lambda-21)^k+(9\lambda^2-42\lambda+77)^k,\end{aligned}$$

$k=1,3,5,7$, which implies Theorem 1.

In conclusion, we wish to thank the referee for his suggestions.

ON THE HOMOGENEOUS CUBIC EQUATION

BY

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1. Let $f(x_1, x_2, \ldots, x_n)$ be a homogeneous cubic form in $x_1, x_2, \ldots, x_n$. Writing

$$x_i = a_i x + b_i y \quad (i = 1, 2, \ldots, n)$$

we get

$$f(x_1, x_2, \ldots, x_n) = x^3 f(a_1, a_2, \ldots, a_n) + x^2 y \, \Sigma \, b_i \frac{\partial f}{\partial a_i} + xy^2 \, \Sigma \, a_i \frac{\partial f}{\partial b_i} + y^3 f(b_1, b_2, \ldots, b_n).$$

The right-hand member reduces to

$$x^3 f(a_1, a_2, \ldots, a_n) + y^3 f(b_1, b_2, \ldots, b_n)$$

if we take

$$x = \Sigma a_i \frac{\partial f}{\partial b_i}, \qquad y = -\Sigma b_i \frac{\partial f}{\partial a_i}$$

We have thus

Theorem 1.—*The equation*

$$f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n) = f(z_1, z_2, \ldots, z_n)$$

has the solution

$$x_i = a_i \Sigma \, a_j \frac{\partial f}{\partial b_j}$$

$$y_i = -b_i \Sigma \, b_j \frac{\partial f}{\partial a_j}$$

$$z_i = a_i \Sigma a_j \frac{\partial f}{\partial b_i} - b_i \Sigma b_j \frac{\partial f}{\partial a_j} \quad (i = 1, 2, \ldots, n).$$

Let us now suppose that

$$f(a_1, a_2, \ldots, a_n) = 0 \text{ and } f(b_1, b_2, \ldots, b_n) = 0.$$

Then Theorem 1 gives

Theorem 2. *If $x_i = a_i$, $x_i = b_i \, (i = 1, 2, \ldots, n)$ are two solutions of $f(x_1, x_2, \ldots, x_n) = 0$, then another solution is given by*

$$x_i = c_i = a_i \Sigma a_j \frac{\partial f}{\partial b_j} - b_i \Sigma b_j \frac{\partial f}{\partial a_j} \qquad (i = 1, 2, \ldots, n).$$

If in Theorem 2, we use the solution $x_i = c_i$ ($i = 1, 2, \ldots, n$) in the place of either of the solutions

$$x_i = a_i,\ x_i = b_i \qquad (i = 1, 2, \ldots, n),$$

say, the latter, then we get by that theorem the solution

$$x_i = a_i \Sigma a_j \frac{\partial f}{\partial c_j} - c_i \Sigma c_j \frac{\partial f}{\partial a_j} \qquad (i = 1, 2, \ldots, n).$$

But

$$\Sigma c_j \frac{\partial f}{\partial a_j} = \Sigma \left\{ a_j\, \Sigma\, a_k \frac{\partial f}{\partial b_k} - b_j\, \Sigma\, b_k \frac{\partial f}{\partial a_k} \right\} \frac{\partial f}{\partial a_j}$$

$$= \left(\Sigma\, a_k \frac{\partial f}{\partial b_k}\right)\left(\Sigma a_j \frac{\partial f}{\partial a_j}\right) - \left(\Sigma b_k \frac{\partial f}{\partial a_k}\right)\left(\Sigma b_j \frac{\partial f}{\partial a_j}\right);$$

and if write

$$p = \Sigma b_k \frac{\partial f}{\partial a_k}, \quad q = \Sigma a_k \frac{\partial f}{\partial b_k}, \quad \phi_j(x_1, \ldots, x_n) = \frac{\partial f}{\partial x_j}, \quad c_i = a_i q - b_i p,$$

and observe that

$$\phi_j(c_1, c_2, \ldots, c_n) = q^2 \phi_j(a_1, \ldots, a_n) - pq\, \Sigma a_i \frac{\partial^2 f}{\partial b_i \partial b_j}$$
$$+ p^2 \phi(b_1, \ldots, b_n) \qquad (j = 1, 2, \ldots, n),$$

then

$$\Sigma a_j \frac{\partial f}{\partial c_j} = q^2 \left(\Sigma a_j \frac{\partial f}{\partial a_j}\right) - pq \left(\sum_{i,j} a_i a_j \frac{\partial^2 f}{\partial b_i \partial b_j}\right) + p^2 \left(\Sigma a_j \frac{\partial f}{\partial b_j}\right).$$

Also, remembering that

$$\Sigma a_i \frac{\partial f}{\partial a_i} = 3f(a_1, \ldots, a_n) = 0,$$

$$\Sigma a_j a_k \frac{\partial^2 f}{\partial b_j \partial b_k} = 2\, \Sigma b_j \frac{\partial f}{\partial a_j} = 2p,$$

the solution

$$x_i = a_i \Sigma a_j \frac{\partial f}{\partial c_j} - c_i\, \Sigma c_j \frac{\partial f}{\partial a_j} \qquad (i = 1, 2, \ldots, n)$$

reduces to

$$x_i = a_i(-2p^2 q + p^2 q) - (a_i q - b_i p)(-p^2) = -b_i p^3$$
$$(i = 1, 2, \ldots, n).$$

This is obviously equivalent to the solution

$$x_i = b_i \qquad (i = 1, 2, \ldots, n).$$

Thus we get

THEOREM 3.—*The three solutions*

$$x_i = a_i$$
$$x_i = b_i$$
$$x_i = c_i = a_i \Sigma a_j \frac{\partial f}{\partial b_j} - b_i \Sigma b_j \frac{\partial f}{\partial a_j} \qquad (i = 1, 2, \ldots, n)$$

of the equation

$$f(x_1, x_2, \ldots, x_n) = 0$$

are such that if the process of obtaining the third from the first two is applied to any two of them, then we get the other.

2. A remarkable special case of Theorem 2 is obtained by taking the cubic form

$$\phi(x_1, \ldots, x_k) - \phi(y_1, y_2, \ldots, y_k)$$

in the $2k$ variables $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ in the place of $f(x_1, \ldots, x_n)$. One set of values of the variables which makes this form zero is obviously

$$x_i = y_i = \lambda_i \qquad (i = 1, 2, \ldots, k)$$

where the λ's are any numbers. Let

$$x_i = a_i,\ y_i = b_i \qquad (i = 1, 2, \ldots, k)$$

be any other solution of the equation

$$\phi(x_1, x_2, \ldots, x_k) = \phi(y_1, y_2, \ldots, y_k).$$

Using these two solutions we get another by Theorem 2. We may state this as

THEOREM 4.—*If* $x_i = a_i,\ y_i = b_i\ (i = 1, 2, \ldots, k)$ *is a solution of the equation*

$$\phi(x_1, x_2, \ldots, x_k) = \phi(y_1, y_2, \ldots, y_k)$$

where ϕ *is a homogeneous cubic form, then*

$$x_i = a_i \Sigma(a_j - b_j)\frac{\partial\phi}{\partial\lambda_j} - \lambda_i \Sigma\lambda_j\left(\frac{\partial\phi}{\partial a_j} - \frac{\partial\phi}{\partial b_j}\right)$$

$$y_i = b_i \Sigma(a_j - b_j)\frac{\partial\phi}{\partial\lambda_j} - \lambda_i \Sigma\lambda_j\left(\frac{\partial\phi}{\partial a_j} - \frac{\partial\phi}{\partial b_j}\right) \quad (i = 1, 2, \ldots, k)$$

is also a solution for all values of $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Take $\phi(x_1, x_2, \ldots, x_k) = \Sigma x_i^3$. Then $\dfrac{\partial\phi}{\partial x_i} = 3x_i^2$, and we get the following

Corollary.—If $x_i = a_i,\ y_i = b_i\ (i = 1, 2, \ldots, k)$ *is a solution of*

$$x_1^3 + x_2^3 + \ldots + x_k^3 = y_1^3 + y_2^3 + \ldots + y_k^3,$$

then

$$x_i = a_i \Sigma(a_j - b_j)\lambda_j^2 - \lambda_i \Sigma\lambda_j(a_j^2 - b_j^2)$$

$$y_i = b_i \Sigma(a_j - b_j)\lambda_j^2 - \lambda_i \Sigma\lambda_j(a_j^2 - b_j^2) \quad (i = 1, 2, \ldots, k)$$

is also a solution for all values of $\lambda_1, \lambda_2, \ldots, \lambda_k$.

For example, taking $k = 2$, $a_1 = 3$, $a_2 = 4$, $b_1 = -5$, $b_2 = 6$ (and writing λ_1 in the place of $2\lambda_1$) we get a two-parameter solution of the equation

$$x_1^3 + x_2^3 = y_1^3 + y_2^3$$

in the form

$$x_1 = 5\lambda_1^2 + 5\lambda_1\lambda_2 - 3\lambda_2^2$$
$$x_2 = 4\lambda_1^2 + 4\lambda_1\lambda_2 + 6\lambda_2^2$$
$$y_1 = -3\lambda_1^2 + 5\lambda_1\lambda_2 + 5\lambda_2^2$$
$$y_2 = 6\lambda_1^2 + 4\lambda_1\lambda_2 + 4\lambda_2^2.$$

3. Let the form $f(x_1, x_2, \ldots, x_n)$ be invariant under the substitution

$$x_j \to x_{i_j} \qquad (j = 1, 2, \ldots, n)$$

where $i_1, i_2, \ldots, i_n$ is some permutation of the numbers 1, 2, 3, ..., n. Then from any solution $x_j = a_j$ $(j = 1, 2, \ldots, n)$ of the equation $f(x_1, \ldots, x_n) = 0$ we get another, namely $x_j = a_{i_j}$ $(j = 1, 2, \ldots, n)$. Using these in Theorem 2 we get

THEOREM 5. *If the cubic form $f(x_1, x_2, \ldots, x_n)$ is invariant under the substitution*

$$x_j \to x_{i_j} \qquad (j = 1, 2, \ldots, n),$$

and if

$$x_j = a_j \qquad (j = 1, 2, \ldots, n)$$

is any solution of

$$f(x_1, x_2, \ldots, x_n) = 0$$

then another solution is given by

$$x_j = a_j\left(\sum_l a_l \frac{\partial f}{\partial a_{i_l}}\right) - a_{i_j}\left(\sum_l a_{i_l} \frac{\partial f}{\partial a_l}\right) \qquad (j = 1, 2, \ldots, n).$$

To complete the proof we have only to remark that

$$\frac{\partial}{\partial a_{il}} f(a_{i_1}, a_{i_2}, \ldots, a_{i_n}) = \frac{\partial}{\partial a_{il}} f(a_1, a_2, \ldots, a_n).$$

Corollary.—If $x_i = a_i$ $(i = 1, 2, \ldots, n)$ is a solution of

$$x_1^3 + x_2^3 + \ldots + x_n^3 = 0$$

and if $i_1, i_2, \ldots, i_n$ is any permutation of 1, 2, ..., n, *then*

$$x_j = a_j \sum a_l a_{i_l}^2 - a_{i_j} \sum a_l^2 a_{i_l} \qquad (j = 1, 2, \ldots, n)$$

is also a solution.

4. A special case of Theorem 5 of some interest is that where in the place of $f(x_1, x_2, \ldots, x_n)$ we take the cubic form

$$f(x_1, x_2, \ldots, x_m) - \phi(y_1, y_2, \ldots, y_n)$$

in $m + n$ variables, f and ϕ being both symmetric in their respective arguments. We then get

THEOREM 6.—*Let* $x_i = a_i$ $(i = 1, 2, \ldots, m)$, $y_i = b_i$ $(i = 1, 2, \ldots, n)$ *be a solution of the equation*

$$f(x_1, x_2, \ldots, x_m) = \phi(y_1, y_2, \ldots, y_n)$$

where f and ϕ are symmetric cubic forms. Let also

$$(a_{k_1}, a_{k_2}, \ldots, a_{k_m}),\ (b_{l_1}, b_{l_2}, \ldots, b_{l_n})$$

be any permutations of $(a_1, a_2, \ldots, a_m)$, $(b_1, b_2, \ldots, b_n)$ *respectively. Then*

$$x_i = a_i \left\{ \sum_{j=1}^{m} a_j \frac{\partial f}{\partial a_{k_j}} - \sum_{j=1}^{n} b_j \frac{\partial \phi}{\partial b_{l_j}} \right\}$$

$$- a_{k_i} \left\{ \sum_{j=1}^{m} a_{k_j} \frac{\partial f}{\partial a_j} - \sum_{j=1}^{n} b_{l_j} \frac{\partial \phi}{\partial b_j} \right\} \quad (i = 1, 2, \ldots, m)$$

$$y_i = b_i \left\{ \sum_{j=1}^{m} a_j \frac{\partial f}{\partial a_{k_j}} - \sum_{j=1}^{n} b_j \frac{\partial \phi}{\partial b_{l_j}} \right\}$$

$$- b_{l_i} \left\{ \sum_{j=1}^{m} a_{k_j} \frac{\partial f}{\partial a_j} - \sum_{j=1}^{n} b_j \frac{\partial \phi}{\partial b_l} \right\} \quad (i = 1, 2, \ldots, n)$$

is also a solution of the equation.

Combining the principles of Theorems 4 and 6 we get

THEOREM 7.—*Let* $(x_1, x_2, \ldots, x_m)$ *be a symmetric cubic form and* $(k_1, k_2, \ldots, k_m)$, $(l_1, l_2, \ldots, l_m)$ *any two permutations of* $(1, 2, \ldots, m)$. *Then the equation*

$$f(x_1, x_2, \ldots, x_m) = f(y_1, y_2, \ldots, y_m)$$

is satisfied by

$$x_i = \lambda_i \sum_j \lambda_j \left(\frac{\partial f}{\partial \lambda_{k_j}} - \frac{\partial f}{\partial \lambda_{l_j}} \right) - \lambda_{k_i} \sum_j (\lambda_{k_j} - \lambda_{l_j}) \frac{\partial f}{\partial \lambda_j}$$

$$y_i = \lambda_i \sum_j \lambda_j \left(\frac{\partial f}{\partial \lambda_{k_j}} - \frac{\partial f}{\partial \lambda_{l_j}} \right) - \lambda_{l_i} \sum_j (\lambda_{k_j} - \lambda_{l_j}) \frac{\partial f}{\partial \lambda_j}$$

$$(i = 1, 2, \ldots, n)$$

for all values of $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Corollary.—If $(k_1, k_2, \ldots, k_m)$, $(l_1, l_2, \ldots, l_m)$ *are two permutations of* $(1, 2, \ldots, m)$ *then the equation*

$$x_1^3 + x_2^3 + \ldots + x_m^3 = y_1^3 + y_2^3 + \ldots + y_m^3$$

has the solution

$$x_i = \lambda_i \sum_j \lambda_j (\lambda_{k_j}^2 - \lambda_{l_j}^2) - \lambda_{k_i} \sum_j (\lambda_{k_j} - \lambda_{l_j}) \lambda_j^2$$

$$y_i = \lambda_i \sum_j \lambda_j \Sigma (\lambda_{k_j}^2 - \lambda_{l_j}^2) - \lambda_{l_i} \sum_j (\lambda_{k_j} - \lambda_{l_j}) \lambda_j^2$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are arbitrary. It can readily be verified that the above solution is non-trivial if $m > 3$ and the permutations are chosen properly.

ON THE HOMOGENEOUS CUBIC EQUATION—GEOMETRICAL APPROACH

BY

A. NARASINGA RAO, *Waltair*

The results of the previous paper by Dr. P. Kesava Menon (pp. 20–24) are best seen in their natural setting in relation to the geometry of cubic varieties, when it will be found that they express in algebraic language certain obvious geometrical properties, and that the known structure of cubic varieties will enable us to obtain other similar results of a deeper character.

We take the set of numbers (a_r), $r = 1, 2, \ldots, n$—not all zero—as the homogeneous co-ordinates of a point in $(n-1)$-dimensional projective space S_{n-1}, the numbers (a_r) and (ka_r), $k \neq 0$, representing the same point. A straight line is the totality of points $x_r = \lambda a_r + \mu b_r$ linearly dependent on two points, while a homogeneous cubic equation $f(x_1, x_2, \ldots, x_n) = 0$ represents a cubic variety. The characteristic property of such an $(n-1)$-dimensional manifold is that any straight line cuts the variety in three points. Hence follow Theorems 2 and 3 of the previous paper, namely that given two solutions of the cubic equation [points (a_r) and (b_r) on the cubic variety], there is associated with them a unique third solution of the form $x_r = \lambda a_r + \mu b_r$, and that the relation between these three is symmetric.

Now on a cubic variety will lie certain flat regions. Thus for $n = 4$, corresponding to a cubic surface in 3-space, we have on it 27 straight lines,* real or imaginary. A plane through any one of these generators cuts the surface again in a conic intersecting the generator twice. Hence we have also on the cubic surface 27 systems of conics, that is to say, solutions $x_r = \lambda^2 a_r + 2\lambda\mu b_r + \mu^2 c_r$ involving two parameters λ, μ in the second degree. Points on such a conic are obtained as linearly dependent on a fixed point of the cubic (*i.e.*, one solution of the equation) and on points of a generator (solutions involving a linear parameter λ). While this is true for all cubic surfaces, Theorem 4 of Dr. Kesava Menon's paper asserts it only for the specialized surface $\phi(x_1, x_2) = \phi(x_3, x_4)$ which has the obvious generator $x_1 = x_3$, $x_2 = x_4$, and for analogous surfaces in hyperspaces of odd dimensions $(n = 2k)$. On the cubic variety given by $\phi(x_1, x_2, \ldots, x_k) = \phi(x_{k+1}, \ldots, x_{2k})$ an obvious generating region is the S_{k-1} defined by

$$x_1 : \ldots x_k : x_{k+1} : \ldots x_{2k} = \lambda_1 : \ldots \lambda_k : \lambda_1 \ldots \lambda_k.$$

* *Vide* Cambridge Tracts No. 27.

The points linearly dependent on a fixed point on the cubic surface and points of S_{k-1} cut the cubic variety again in a V_{k-1}^2 and hence give solutions involving quadratically k parameters. This is the content of Theorem 4.

We know* that on the cubic surface in 3-space lie 27 straight lines which intersect in 135 points and form 45 tritangent planes,—planes whose intersections with the cubic surface reduce to three lines. Accordingly we deduce the existence of 27 solutions of the homogeneous cubic equation, of the type $x_r = \lambda a_r + \mu b_r$ which are solutions for all values of λ and μ, and also of 45 triads of solutions (a_r), (b_r), (c_r) such that $x_r = \lambda a_r + \mu b_r$, $x_r = \mu b_r + \nu c_r$, and $x_r = \nu c_r + \lambda a_r$ are all solutions for every value of λ, μ, ν. We know also that at most six of the 27 lines can be skew to one another. This translates into the result that we can find at most six sets of solutions each involving a parameter linearly, such that no two sets have a common solution. In the same manner one may interpret the double-six and other configurations, on the surface.

The cubic represented by $f(x_1, x_2) = \phi(x_3, x_4)$ represents a specialized form, for by a suitable linear transformation both f and ϕ (which involve only two variables each) may be reduced to the sum of two cubes, leading to the canonical form $x_1^3 + x_2^3 = x_3^3 + x_4^3$ or its equivalent $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$. The general cubic form in four variables, on the other hand, can be represented only as the sum of the cubes of *five* linear forms. Such an "equianharmonic cubic surface", $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$ is carried over into itself by the cyclic homologies of period 3 given by

$$x_1' : x_2' : x_3' : x_4' = \epsilon x_1 : x_2 : x_3 : x_4, \qquad \epsilon^3 = 1$$

and the others obtained by permutations of x_1, x_2, x_3, x_4. The Hessian breaks up into the faces of the fundamental tetrahedron.

If the reference tetrahedron is real, the equianharmonic surface $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$ contains 3 real lines and 7 real tritangent planes.† If 3 co-planar lines on a cubic surface are concurrent, this point of concurrence is called an Eckardt point. While the general non-singular cubic surface has no Eckardt point, the equianharmonic cubic has 6 real Eckardt points which are the vertices of a plane quadrilateral, being the intersections of the 6 edges of the fundamental tetrahedron with the plane $x_1 + x_2 + x_3 + x_4 = 0$. The three real lines on the surface are obviously $x_r = -x_s$, $x_t = -x_u$ $(r, s, t, u = 1, 2, 3, 4)$, while the Eckardt points have co-ordinates of the type $(0, 0, 1, -1)$, etc. The three generators through $(0, 0, 1, -1)$, are given by $x_1 + \epsilon x_2 = 0$, $x_3 + x_4 = 0$, $(\epsilon^3 = 1)$ so that one is real and the others are conjugate imaginary. These results may be interpreted in terms of solutions of the cubic equation.

* Segre, "The Non-Singular Cubic Surface," Oxford, 1942.

† Segre, "The Non-Singular Cubic Surface," pp. 128–29.

It is worth noting that the co-ordinates of the third point R of intersection of the line joining P and Q on the surface, are dependent rationally on those of P, Q, and the coefficients in the equation, so that if these are rational numbers, so are the co-ordinates of R. By choosing rational points on the generating region we may get solutions of any homogeneous cubic equation with integer coefficients, involving several parameters with integral coefficients, and of the first or second degree in the parameters.

ON $\tau(p)$ MOD 49

A careful study of my "Table of Values of $\tau(n)$" for values of n up to 400, has led me to the

Conjecture: If p be a prime of the form $7q+r$, $r = 3, 5$ or 6 then

$$\tau(p)/7 \equiv r - 1 - [3/r] - 2q \pmod 7, \tag{1}$$

where $\tau(n)$ is Ramanujan's function and $[x]$ denotes as usual the greatest integer in x.

In case the above conjecture is true, it would follow that

$$\tau(p_1^a) \equiv \tau(p_2^a) \pmod{49}, \; a \geqslant 1; \tag{2}$$

where the primes p_1 and p_2 are congruent mod 49 and

$$\left(\frac{p_1}{7}\right) = -1.$$

The conjecture (2) is due to Lehmer.

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RECURRING SEQUENCES

BY

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We consider the general recurring sequence (W) of order $s \geqslant 2$ defined by

$$W_n = a_1 W_{n-s} + a_2 W_{n-s+1} + \ldots + a_s W_{n-1} \quad (n = 0, \pm 1, \pm 2, \ldots) \quad (1)$$

with arbitrary complex $a_1 \neq 0$, $a_2, \ldots, a_s$, $W_0, W_1, \ldots, W_{s-1}$, and also the special case (U) with the same $a_1, \ldots, a_s$ and

$$U_0 = U_1 = \ldots = U_{s-2} = 0, \; U_{s-1} = 1. \quad (2)$$

The following statement holds.

$$\begin{aligned} W_{m+n} = {} & a_1 W_{m-1} U_n \\ & + (W_{m+s-2} - a_s W_{m+s-3} - a_{s-1} W_{m+s-4} - \ldots - a_3 W_m) U_{n+1} \\ & + (W_{m+s-3} - a_s W_{m+s-4} - a_{s-1} W_{m+s-5} - \ldots - a_4 W_m) U_{n+2} \\ & + \ldots \\ & + (W_{m+1} - a_s W_m) U_{n+s-2} \\ & + W_m U_{n+s-1}. \end{aligned} \quad (3)$$

Indeed, (3) can be written as follows:

$$\begin{aligned} W_{m+n} = W_{m+s-1} U_n - W_{m+s-2} a_{s-1} U_n - W_{m+s-3} a_{s-2} U_n - \ldots - W_m a_2 U_n \\ + W_{m+s-2} U_{n+1} - W_{m+s-3} a_s U_{n+1} - \ldots - W_m a_3 U_{n+1} \\ + W_{m+s-3} U_{n+2} - \ldots - W_m a_4 U_{n+2} \\ + \ldots \\ + W_m U_{n+s-1}. \end{aligned} \quad (3')$$

Summing (3') by columns we easily see that (3') holds for $n = 0, 1, 2, \ldots, s-1$, whence (3') holds generally.

Putting in (3), $W \equiv U$, $m = (k-1)n + r$, we have :

$$\begin{aligned} U_{kn+r} = {} & a_1 U_{(k-1)n+r-1} U_n + A_1 U_{n+1} + A_2 U_{n+2} + \ldots + A_{s-2} U_{n+s-2} \\ & + U_{(k-1)n+r} U_{n+s-1}, \end{aligned} \quad (4)$$

where the A's are polynoms in a's and U's.

In particular, for $s \geqslant 4$, $k = 2$, $r = 2, 3, \ldots, s-2$, (4) becomes:

$$U_{2n+r} = B_1(r) U_{n+1} + B_2(r) U_{n+2} + \ldots + B_{s-2}(r) U_{n+s-2}, \quad (5)$$

where the B's are polynoms in a's and U's.

Putting in (3), $m = W_0 = 0$, we have:

$$W_n = c_0 U_n + c_1 U_{n+1} + c_2 U_{n+2} + \ldots + c_{s-2} U_{n+s-2}, \quad (6)$$

where the c's are polynoms in a's and W's.

In what follows, $a_1, \ldots, a_s$ and $W_0, \ldots, W_{s-1}$ are supposed to be integers. Then evidently all W_n and $a_1^n W_{-n}$ with $n \geq 0$ are integers. We shall say that a fraction P/Q, where P, Q are integers and $Q \neq 0$, is divisible by an integer p, if P is divisible by p.

It is known that (U), being a recurring sequence, is periodic for every modul. From this fact and from the fact that $U_0 = U_1 = \ldots = U_{s-2} = 0$ we deduce that

For every positive integer p there exists a positive integer n such that p is a common divisor of $U_n, U_{n+1}, \ldots, U_{n+s-2}$. (7)

Every common divisor of

$$U_n, U_{n+1}, U_{n+2}, \ldots, U_{n+s-2}$$

also divides

$$U_{kn}, U_{kn+1}, U_{kn+2}, \ldots, U_{kn+s-2}. \quad (8)$$

Indeed, putting in (4) $r = 0, 1, 2, \ldots, s-2$ successively, we see that (8) holds for k, if it holds for $k-1$. But (8) holds evidently for $k = 1$, hence (8) holds for every $k \geq 1$. From the fact that (U) is periodic backwards also, we deduce that (8) also holds for $k < 1$.

From (5), we have :

For $s \geq 4$, *every common divisor of*

$$U_{n+1}, U_{n+2}, \ldots, U_{n+s-2}$$

also divides

$$U_{2n+2}, U_{2n+3}, \ldots, U_{2n+s-2}. \quad (9)$$

From (6) and (8) we have:

If $W_0 = W_1 = \ldots = W_t = 0$, *then every common divisor of*

$$U_n, U_{n+1}, \ldots, U_{n+s-2}$$

also divides

$$W_{kn}, W_{kn+1}, \ldots, W_{kn+t} \quad (10)$$

Definition.—The least positive integer n, such that $U_n, U_{n+1}, \ldots, U_{n+s-2}$ are divisible by a positive integer p, shall be called the *rank of apparition of p* in (U).

Every common divisor p of $W_m, U_n, U_{n+1}, \ldots, U_{n+s-2}$ *also divides* W_{m-n}. (11)

Indeed, by (8), p also divides $U_{-n}, U_{-n+1}, \ldots, U_{-n+s-2}$. Hence, putting $-n$ instead of n in (3), we obtain (11).

If n is the rank of apparition of p in (U), then every number N, such that

$$U_N, U_{N+1}, \ldots, U_{N+s-2} \qquad (12)$$

are divisible by p, is a multiple of n.

Indeed, supposing $N = kn + r$, where $0 < r < n$, we see by (8) and (11) that $U_r, U_{r+1}, \ldots, U_{r+s-2}$ are divisible by p, which is impossible, since $0 < r < n$ and n is the rank of apparition of p in (U).

LEMMA.—*Let n be the rank of apparition of p in (U), and let P be the length of the period modulo p. If* $n < P/(d-1)$, *where* $d \geqslant 2$, *then* $n \leqslant P/d$.

Indeed, supposing $n = (P/d) + r$, where $0 < r < P/d(d-1)$, we have by (8): $U_{dr+t} \equiv U_{P+dr+t} = U_{dn+t} \equiv 0 \pmod p$, that is, $U_{dr+t} \equiv 0 \pmod p$ for $t = 0, 1, 2, \ldots, s-2$, which is impossible by the meaning of a rank of apparition, since $dr = (d-1)r + r < (P/d) + r = n$, that is, $dr < n$.

If n is the rank of apparition of p in (U), and if P is the length of the period modulo p, then n is a divisor of P. (13)

Indeed, if n is not a divisor of P, we can, by repeated use of the lemma, deduce that $n < P/d$ for every positive integer $d \geqslant 2$, that is $n \leqslant 0$, which is impossible by the definition of the rank of apparition.

NOTES AND DISCUSSIONS

Sequences satisfying $a_1 = 1$ and $(r+1)^{-1} a_r \leqslant a_r - a_{r+1} \leqslant r^{-1} a_{r-1}$ for $r > 1$.

Let $f(x, c)$ denote the number of integers $\leqslant x$ and free of prime-divisors $> x^c$. In a paper* published elsewhere I have proved among other things that

(A) a continuous function $\phi(c)$ exists such that, for any fixed c,

$$f(x, c) \sim x \phi(c),$$

(B) $\phi(c) = 1$ for $c \geqslant 1$, and $c \phi'(c) = \phi\left(\frac{c}{1-c}\right)$ for $0 < c < 1$, and

(C) $0 < \phi(c) < A_1 \left\{\Gamma\left(\frac{c+1}{c}\right)\right\}^{-1}$, for $c > 0$; where A_1 is a constant.

Now it easily follows from (B) and (C) that the sequence $a_r = \phi\left(\frac{1}{r}\right)$, where $r = 1, 2, 3, \ldots.$ satisfies:—

(D) $a_1 = 1$, and $(r+1)^{-1} a_r \leqslant a_r - a_{r+1} \leqslant r^{-1} a_{r-1}$ for $r > 1$, and

(E) $a_r \geqslant 0$ for $r \geqslant 1$.

It is of obvious interest to see whether the full force of (B) is necessary to prove (C), or whether something like (D) would suffice. In fact, when the results (A), (B) and (C) were substantially announced at the Waltair Conference (Dec. 24, 1947) of the Indian Mathematical Society, it was contended that (C) was wrong, and that a proof could be given of the existence of a number k such that $ra_r > k > 0$ for every r. Of course any such proof based on (D) and (E) must be fallacious, as can be seen from the obvious fact that $a_r = \{\Gamma(r)\}^{-1}$ satisfies (D) and (E) and yet $ra_r \to 0$ as $r \to \infty$. It is also obvious that $a_r = \frac{\lambda}{r} + \frac{(1-\lambda)}{\Gamma(r)}$, where $0 < \lambda \leqslant 1$ satisfies (D) and (E) and that $ra_r \to \lambda$ as $r \to \infty$.

Now one would be interested to know if (D) implies (E), and if so, whether orders of magnitude between $O\left(\frac{1}{r}\right)$ and $O\left(\frac{1}{\Gamma(r)}\right)$ are possible for a sequence satisfying (D); for instance it is easily verified that the sequence $a_1 = 1$, $a_r = \{\Gamma(r-1)\}^{-1}$ for $r > 1$ does not satisfy (D). So one is led to the following conjecture, which happens to be true:

If any sequence (a_r) satisfies (D); i.e., if

(1) $a_1 = 1$, *and* $(r+1)^{-1} a_r \leqslant a_r - a_{r+1} \leqslant r^{-1} a_{r-1}$ *for* $r > 1$,

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* A paper entitled *On the number of integers $\leqslant x$ and free of prime-divisors $> x^c$ and a problem of S. S. Pillai*,—*Duke Mathematical Journal*, Vol. 16, No. 1, March 1949.

then

(2) $a_r \geqslant 0$ *for every* r, [*i.e.*, (E) *is true*],

and

(3) (*a*) ra_r *is a monotonic non-increasing sequence for* $r \geqslant 2$, *and either*

(*b*) $ra_r \to k > 0$ *as* $r \to \infty$,

or

(*c*) $a_r \leqslant \frac{1}{\Gamma(r)}$ *for* $r \geqslant 1$.

Proof.—From the left-hand inequality in (1) follows

(4) $ra_r \geqslant (r+1)\, a_{r+1}$ for $r > 1$, which proves (3*a*).

Hence

(5) $ra_r \to l \leqslant 2a_2$ (*including* $l = -\infty$ *possibly*) *as* $r \to \infty$.

Now

(6) from (4) and (5), (2) follows in case $l > 0$.

Now, subtracting $a_r r^{-1}$ from the two sides of the right-hand inequality in (1), we have

(7) $(r-1)\, a_r - ra_{r+1} \leqslant a_{r-1} - a_r$ for $r > 1$.

Summing such inequalities from r to R, where $\mathrm{R} > r > 1$, follows

(8) $(r-1)\, a_r - \mathrm{R}a_{\mathrm{R}+1} < a_{r-1} - a_{\mathrm{R}}$ for $\mathrm{R} > r > 1$

i.e., $(r-1)\, a_r \leqslant a_{r-1} + \mathrm{R}a_{\mathrm{R}+1} - \mathrm{R}a_{\mathrm{R}}\mathrm{R}^{-1}$

$\leqslant a_{r-1} + \mathrm{R}a_{\mathrm{R}+1} - (\mathrm{R}+1)\, a_{(\mathrm{R}+1)}\mathrm{R}^{-1}$, *by* (4).

Hence

$$(r-1)\, a_r \leqslant a_{r-1} + (\mathrm{R}+1)\, a_{\mathrm{R}+1}\left(\frac{\mathrm{R}}{\mathrm{R}+1} - \frac{1}{\mathrm{R}}\right)$$

and also, by (4)

$$(\mathrm{R}+1)\, a_{\mathrm{R}+1}\left(1 - \frac{1}{r}\right) \leqslant ra_r\left(1 - \frac{1}{r}\right) = (r-1)\, a_r, \textit{ for } \mathrm{R} > r > 1;$$

and from these, it follows

(9) $a_{r-1} \geqslant (\mathrm{R}+1)\, a_{\mathrm{R}+1}\left[\frac{1}{\mathrm{R}+1} + \frac{1}{\mathrm{R}} - \frac{1}{r}\right]$ *for* $R > r > 1$.

Keeping r fixed and making $\mathrm{R} \to \infty$ in this, follows

(10) $a_{r-1} \geqslant 0$ *for* $r > 1$, *in case* $l \leqslant 0$.

Now, (6), (10), and (1) prove (2). From this and (4), it follows either

(11) $ra_r \to k > 0$; which is (3*b*);

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or

(12) $ra_r \to 0$; in which case (8) gives, by making $R \to \infty$,

(13) $(r-1)\,a_r \leqslant a_{r-1}$ for $r > 1$;

and hence

(14) $a_r \leqslant \dfrac{1}{\Gamma(r)}$ *for* $r > 1$, *and hence also obviously for* $r \geqslant 1$;

which is (3*c*).

Corollary.—If a sequence (a_r) satisfies (1), and $a_r > \dfrac{1}{\Gamma(r)}$ for some then

(15) a positive number k exists such that $a_r > \dfrac{k}{r}$ for every r.

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Note.—Simultaneously with me Dr. V. Ganapathy Iyer proved the slightly less precise result: If any sequence (a_r) satisfies (D) and (E), and "no two positive numbers μ and λ exist such that $a_r < \mu\,\{\Gamma(r-1-\lambda)\}^{-1}$ for all sufficiently large r," then (15) is true. It is seen from the above that, in order to ensure (15), it is sufficient if (D) holds, and either ra_r does not tend to zero, or $a_r > \dfrac{1}{\Gamma(r)}$, for some r.

On the Circumcircles of Rectangles Inscribed in a Triangle*

Given a triangle ABC, we have three systems of rectangles inscribed in ABC, each associated with a side (or altitude) of the triangle. The circumcircles of these rectangles form also three systems whose properties are discussed here. The centres of the members of each system are collinear and the three loci of centres are concurrent at the symmedian point. The cosine circle itself is a common member of the three systems. The members of each system have double contact with a central conic. The properties of these systems of circles are similar to those of the systems studied in my earlier paper "On a system of circles associated with a symmedian of a triangle".

Let ABC be a triangle; A_1, B_1, C_1, the middle points of its sides; D, E, F, the feet of the altitudes; O the orthocentre and L, M, N, the middle points of AD, BE, CF. We have then

THEOREM I.—*If* P *is any point on* AD—*the altitude through* A—*and if points are taken* X, X_1, *on* BC, Y *on* CA *and* Z *on* AB *such that*

$$\frac{DP}{DO} = \frac{DX}{DB} = \frac{DX_1}{DC} = \frac{AY}{AC} = \frac{AZ}{AB} = \lambda,$$

the points XX_1YZ *are concyclic.*

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* Vide *The Mathematics Student*, Vol. XV, 1947, pp. 101–2.

If this circle cuts AC, AB *again in* Y_1, Z_1 *then*

(1) XX_1YZ *is a rectangle*

and (2) *the centre of this circle is a point* T *on* A_1L *such that* $LT = \lambda.LA_1$.

If the point P varies on AD, we get a system of circles whose centres lie on the line A_1L. It may be noted that the circles on BC and AD as diameters belong to this system, being the limiting cases when the rectangle tends to coincidence with BC and AD respectively.

With L as origin and A_1L as the x-axis, the equation to the circle of Theorem 1 is of the form

$$(x - \lambda p)^2 + y^2 = \lambda^2 q^2 + (1 - \lambda)^2 r^2,$$

where $p = A_1L$, $q = \frac{1}{2}$ BC, and $r = \frac{1}{2}$ AD.

The six points in which the cosine circle cuts the sides satisfy the conditions of Theorem 1. Hence we have:

THEOREM II.—*All the three systems of circles have the cosine circle as a common member.*

Since the circles in Theorem 1 are of the form $\lambda^2 S_1 + \lambda S_2 + S_3 = 0$, where S_1, S_2, S_3 are respectively of degrees 0, 1 and 2 in x and y, the envelope E of the system of circles, namely $S_2{}^2 = 4\, S_1 S_3$, is of the second degree. The equation to the system of circles can be put in the form

$$E + \{px - r^2 - \lambda(p^2 - q^2 - r^2)\}^2 = 0.$$

The E-conic is a central conic with A_1L as one of its principal axes. We have then

THEOREM III.—*All the members of a system of circles have double contact with a central conic.*

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A Curvature Property of a Certain Class of Curves

For the curve

$$x^m y^n = a^{m+n}, \quad (m \neq 0,\ n \neq 0,\ m + n \neq 0),$$

we have the relation

$$\left\{\left(\frac{d\rho}{ds}\right)^2 - \lambda\rho\,\frac{d^2\rho}{ds^2} + 3\lambda\mu + \lambda^2\right\}\left\{\left(\frac{d\rho}{ds}\right)^2 - \mu\rho\,\frac{d^2\rho}{ds^2} + 3\lambda\mu + \mu^2\right\} + \lambda\mu(\lambda - \mu)^2 = 0,$$

where $\lambda = \dfrac{m + 2n}{m + n}$, $\mu = \dfrac{2m + n}{m + n}$.

It is verified that

$$\rho = k\,(\cos\omega)^{-\lambda}\,(\sin\omega)^{-\mu},$$

where k is a constant and ω is the angle between the x-axis and the perpendicular from the origin on the tangent. The above relation is deduced by using the formulæ

$$\frac{dp}{ds} = \frac{1}{\rho}\,\frac{dp}{d\omega}, \qquad \rho\,\frac{d^2p}{ds^2} = \frac{d}{d\omega}\left(\frac{1}{\rho}\,\frac{dp}{d\omega}\right).$$

The case $m = -1$, $n = 2$ gives for the parabola, $y^2 = 4ax$, the known result

$$3p\,\frac{d^2p}{ds^2} - \left(\frac{dp}{ds}\right)^2 - 9 = 0.$$

Similarly $m = 1$, $n = 1$ gives for the rectangular hyperbola $xy = c^2$,

$$3p\,\frac{d^2p}{ds^2} - 2\left(\frac{dp}{ds}\right)^2 - 18 = 0.$$

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Annamalainagar, T. K. GURUSWAMY.

A Note on a Property of the Median

PROPERTY.—*The mean deviation is the least when measured from the median.*

A proof of this result is found in almost every text-book on statistics. The following proof is of interest as it holds good equally for the continuous and the discontinuous distributions, being based on the properties of Stieltjes Integrals. In fact, even the existence of the frequency function has not been assumed.

Proof.—The mean deviation about an arbitrary point a is

$$\delta(a) = \int_{-\infty}^{a} |\,x - a\,|\,.\,dF(x),$$ where $F(x)$ is the distribution function.

$$= \int_{-\infty} (a - x)\,dF + \int_{a}^{\infty} (x - a)\,dF$$

$$\therefore\ \delta(a+h) - \delta(a) = \int_{-\infty}^{a+h} (a + h - x)\,dF - \int_{-\infty}^{a} (a - x)\,dF + \int_{a+h}^{\infty} (x - a - h)\,dF - \int_{a}^{\infty} (x - a)\,dF$$

$$\therefore\ \frac{\delta(a+h) - \delta(a)}{h} = \int_{-\infty}^{a} dF - \int_{a}^{\infty} dF + 2\int_{a}^{a+h} \frac{a + h - x}{h}\,dF,$$

Now the lower and upper bounds of $\frac{a+h-x}{h}$ in $(a, a+h)$ are 0 and 1 and therefore*

$$\int_a^{a+h} \frac{a+h-x}{h}\, dF = \lambda \,. [F(a+h) - F(a)], \text{ where } 0 \leqslant \lambda \leqslant 1.$$

$$\therefore \lim_{h \to 0} \frac{\delta(a+h) - \delta(a)}{h} = A(a) + 2\lambda [F(a+h) - F(a)]$$

where $A(a) = \int_{-\infty}^{a} dF - \int_{a}^{\infty} dF.$

If $F(x)$ has no 'saltus' at a, $\lim_{h \to 0} [F(a+h) - F(a)] = 0$

$\therefore \lim_{h \to 0} \frac{\delta(a+h) - \delta(a)}{h}$ exists and is equal to $A(a)$.

$\therefore \delta'(a) = A(a)$.

But for $\delta(a)$ to be a maximum or a minimum, we must have $\delta'(a) = 0$, when it exists.

$$\therefore \int_{-\infty}^{a} dF - \int_{a}^{\infty} dF = A(a) = \delta'(a) = 0.$$

$\therefore$ the median is at a.

To show that $\delta(a)$ is a *minimum* and *not* a maximum, we notice that $\int_{-\infty}^{a} dF$ being the frequency from $-\infty$ to a must increase with a and similarly $\int_{a}^{\infty} dF$ must decrease with a so that $\delta'(a)$ must increase with a, showing that $\delta(a)$ is a *minimum*.

If F (x) has a 'saltus' at a, it can not† have any other saltus in a small interval $(a-k, a+k)$ so that $(a-k)$ is a continuity point of $F(x)$. Therefore, if we start with the mean deviation about $(a-k)$, we shall arrive at the result that it is a minimum when $(a-k)$ is the median of the distribution and as k can be taken arbitrarily small, the same result follows.

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* H. Cramer, *Mathematical Methods of Statistics*, § 6·3 and § 7.1.4.

† Follows from § 6·2 of the same book.

Note on "Calculation of Cumulants"

Cumulants are defined by the identity in t

$$\phi(t) = \exp\left\{k_1 t + k_2 \frac{t^2}{2!} + \ldots + k_r \frac{t^r}{r!} + \ldots\right\}$$

$$= 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots + \mu'_r \frac{t^r}{r!} + \ldots \tag{1}$$

where $\mu_1', \mu_2', \ldots$ are the moments of a distribution about any arbitrary point.

Thus, whereas μ_r' is the coefficient of $\frac{t^r}{!}$ in $\phi(t)$, k_r is the coefficient of

in $\log \phi(t)$, if an expansion in power series exist.

We differentiate both sides of the equation (1) with respect to k_j, then

$$\frac{t^j}{j!}\left[1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots + \mu'_r \frac{t^r}{r!} + \ldots\right\}$$

$$= \frac{t^j}{j!}\frac{\partial \mu_j'}{\partial k_j} + \frac{t^{j+1}}{(j+1)!}\frac{\partial \mu_j'{}_{+1}}{\partial k_j} + \ldots \tag{2}$$

Equating the coefficients of t^r, we get from equation (2)

$$\frac{\partial \mu_r'}{\partial k_j} = \binom{r}{j} \mu_r'{}_{-j} \tag{3}$$

Now

$$d\mu'_r = \frac{\partial \mu_r'}{\partial k_r} dk_r + \frac{\partial \mu_r'}{\partial k_{r-1}} dk_{r-1} + \ldots + \frac{\partial \mu_r'}{\partial k_1} dk_1 \tag{4}$$

From (3) we get

$$\left.\begin{aligned}\frac{\partial \mu_r'}{\partial k_r} &= \binom{r}{r} \mu'_0 = 1, \ \frac{\partial \mu_r'}{\partial k_{r-1}} = \binom{r}{r-1}\mu_1', \\ \frac{\partial \mu_r'}{\partial k_{r-2}} &= \binom{r}{r-2}\mu_2', \ldots, \frac{\partial \mu_r'}{\partial k_1} = \binom{r}{1}\mu_r'{}_{-1}.\end{aligned}\right\} \tag{5}$$

Therefore, (4) becomes

$$d\mu_r' = dk_r + \binom{r}{1}\mu_1' dk_{r-1} + \binom{r}{2}\mu_2' dk_{r-2}$$
$$+ \ldots + \binom{r}{r-1} \mu'_{r-1} dk_1. \tag{6}$$

Now we have obtained a differential equation with the help of equations (4) and (5). Therefore integrating equation (6) we get

$$\int d\mu_r' = \int dk_r + \binom{r}{1}\int \mu_1' dk_{r-1} + \ldots$$
$$+ \binom{r}{j}\int \mu_j' dk_{r-j} + \ldots + \binom{r}{r-1}\int \mu'_{r-1} dk_1. \tag{7}$$

When r is even and $j = \frac{r}{2}$ or when r is odd and $j = \frac{r-1}{2}$, μ_j' contains terms involving k_{r-j} and k's with higher suffixes. In order to avoid the recurring of the terms already integrated we neglect the terms in all the following integrals which involve k's whose suffixes are higher than dk_{r-j}, *i.e.*, the k's involved in μ_j' whose suffixes are $r-j+1$, $r-j+2$, ...

Similarly the form of equation (7) for k's in terms of μ''s becomes

$$\int dk_r = (-1)^{\rho-1}(\rho-1)!\left\{\int d\mu_r' + \binom{r}{1}\int k_1\, d\mu'_{r-1} + \binom{r}{2}\int k_2\, d\mu'_{r-2} + \ldots + \binom{r}{r-1}\int k_{r-1}\, d\mu_1'\right\} \quad (8)$$

where ρ is equal to the sum of powers of μ''s in every integrated μ'.

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S. M. HASNAIN JAFFRI.

COLLEGIATE SECTION

A Note on Particular Integrals of Linear Differential Equations with Constant Coefficients

With the usual notation,

$$\frac{1}{f(D)}\, e^{mx} = \frac{1}{f(m)}\, e^{mx}, \text{ if } f(m) \neq 0,$$

while if $f(D) = (D - m)\, \phi(D)$, where $\phi(m) \neq 0$,

$$\frac{1}{f(D)}\, e^{mx} = \frac{xe^{mx}}{\phi(m)}, \text{ worked out in the usual way.}$$

The object of this note is to point out a simple way in which this can be remembered, and generalized. We have,

$$f'(D) = (D - m)\, \phi'(D) + \phi(D).$$

Hence $\quad f'(m) = \phi(m)$, thus giving

$$\frac{1}{f(D)}\, e^{mx} = \frac{xe^{mx}}{f'(m)}.$$

If $f'(m) = 0$, but not $f''(m)$, the argument can be repeated and we get

$$\frac{1}{f(D)}\, e^{mx} = \frac{x^2\, e^{mx}}{f''(m)}, \text{ and so on.}$$

The same type of formula obviously holds good if we have $\sin mx$ or $\cos mx$ instead of e^{mx}.

Example—

$$\frac{1}{(D^2 + 4)^2} \sin 2x = \frac{1}{(D - 2i)^2\ (D + 2i)^2} \sin 2x$$

$$= \text{coefficient of } i \text{ in } \frac{x^2 e^{2ix}}{f''(2i)}, \text{ where } f(D) = (D^2 + 4)^2$$

$$= -\frac{x^2 \sin 2x}{32}.$$

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JAGANNATH MITAL

A Note on the Centre of Pressure

The following is an empirical rule on the centre of pressure of plane areas immersed in a heavy homogeneous liquid. The rule is analogous to Dr. Routh's rule on Moments of Inertia and has not been pointed out in any text-book, as far as I am aware.

1. *A parallelogram with one side in the surface*

The centre of gravity lies on the line joining the midpoints of the side in the surface and the opposite side and is at $\frac{1}{2}$ this distance from the surface. The centre of pressure lies on the same line and is at $\frac{1+1}{2+1} = \frac{2}{3}$ this distance from the surface.

2. *Triangular lamina with one side in the surface.*

The C. G. lies on the median to this side and is at $\frac{1}{3}$ this length from the surface. The C. P. lies on the same line and is at $\frac{1+1}{3+1} = \frac{1}{2}$ this distance from the surface.

3. *Triangular lamina with a vertex in the surface and the opposite side horizontal*

The C. G. lies on the median through the vertex in the surface and is at $\frac{2}{3}$ this distance from the vertex. The C. P. lies on this median and is at $\frac{2+1}{3+1} = \frac{3}{4}$ this distance from the vertex.

The rule in the above cases is:—

"Increase the numerator and denominator of the depth ratio of the centre of gravity by 1 each; this gives the depth ratio of the centre of pressure."

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On $\int \frac{Ax^2 + B}{x^4 + 2ax^2 + c^2} dx$

A more general integral which includes all the types given by Mr. C. S. Venkataraman,* is $\int \frac{Ax^2 + B}{x^4 + 2ax^2 + c^2} dx$. The following method avoiding partial fractions, has been used by me in my class-room lectures.

* *Mathematics Student*, Vol. XV, Nos. 3 & 4, pp. 105–07.

Put $x^2 + c = z_1 x$ or $x + \frac{c}{x} = z_1$

then $1 - \frac{c}{x^2} = \frac{dz_1}{dx}$.

Hence $x^4 + 2ax^2 + c^2 = \{z_1^2 + 2(a-c)\} x^2$

$$\int \frac{x^2 - c}{x^4 + 2ax^2 + c^2} dx = \int \frac{dz_1}{z_1^2 + 2(a-c)}$$

Changing c to $-c$, we have $z_2 = x - \frac{c}{x}$

and $$\int \frac{x^2 + c}{x^4 + 2ax^2 + c^2} dx = \int \frac{dz_2}{z_2^2 + 2(a+c)}.$$

Now $Ax^2 + B$ may be put in the form $p(x^2 + c) + q(x^2 - c)$

where $2p = A + \frac{B}{c}$, $2q = A - \frac{B}{c}$.

Hence the given integral is reduced to

$$\tfrac{1}{2}\left(A - \frac{B}{c}\right) \int \frac{dz_1}{z_1^2 + 2(a-c)} + \tfrac{1}{2}\left(A + \frac{B}{c}\right) \int \frac{dz_2}{z_2^2 + 2(a+c}$$

which can be evaluated from standard forms.

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Books for Your Library ?

Certainly ! The *Mathematics Student* will gladly help you in making a selection suitable for your needs so that your library, whether private or belonging to an institution, may be well equipped in the manner you desire.

Advanced treatises in mathematics are like mining shafts, each going deep down into the world of abstract thought and enabling you to handle beautiful gems, but not giving you any view of other mines unless they happen to be connected with yours. So before we leave Sunlight and Fresh Air, let us take a flight over the whole domain of mathematical thought and get an airman's view of this peculiar activity of the human species and the kind of creatures engaged therein. You can select no better pilot for this than the versatile Prof. E. T. Bell who is equally at home in mathematics, number magic and knowledge of men and things, and whose magic pen recreates the men and situations of the distant past in a manner which will fascinate you and keep you from your dinner. The twin publications

E. T. BELL : " Men of Mathematics "

E. T. BELL : " Development of Mathematics "

both published by Simon and Schuster, New York, as long ago as 1937, will be found to be as exciting as a detective novel and should be in every library. No better books could be recommended for those who wish to see mathematics not as a dead examination subject, but as a living and growing reality built up gradually by men and women like ourselves, with our ambitions and our weaknesses.

An equally remarkable book by another well-known mathematician is

G. POLYA: "How to Solve It," Stanford University, Princeton University Press, pp. 204, 1945.

This is a careful and clear analysis of the methodology involved in the solution of problems in general and mathematical problems in particular. This is not a haphazard process, but depends on a clear understanding of the problem, the devising of a suitable plan, the carrying out of the plan, and lastly the verification that it works and the final survey of the result. The experienced teacher will find in this book much that he has empirically discovered for himself regarding the methodology of attacking problems, presented with a clarity and force for which he should be truly thankful; and the students will find that even for groping in the dark there are rules which make for success! Polya's book must be in every library.

Before concluding, I should like to refer to a series of little books on modern mathematics, of pocket size, and very attractively got up, which seek to explain the essential ideas of the topics they deal with so as to be clear even to the lay reader, without omitting the mathematics entirely. They will be ideal for a first study of the subject, and are warmly recommended for purchase, if copies are still available. The titles of the three books in the series are:

Non-Euclidean Geometry,

Galois and the Theory of Groups

The Einstein Theory of Relativity.

They are published by H. G. L. R. Lieber, 258, Clinton Ave, Brooklyn, N.Y.

A. N. RAO.

An Instructive Problem and its Solution

From any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *normals are drawn to the curve. Show that the locus of the orthocentre of the triangle formed by joining their feet is the ellipse* $a^2x^2+b^2y^2=\left(\frac{a^4+b^4}{a^2-b^2}\right)^2$. (Askwith, *Analytical Geometry*, p. 179, Ex. 17).

Let $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$

Let Q, R, S be the feet of normals from P, and let Q be the point θ.

Then PQ is $ax \sec\theta - by \operatorname{cosec}\theta = a^2-b^2$.

So the equation of RS is $\frac{x\cos\theta}{a^3}-\frac{y\sin\theta}{b^3}+\frac{1}{a^2-b^2}=0.$

Let Σ be the envelope of RS and evidently it touches QR and QS also. $\Sigma \equiv a^6l^2+b^6m^2-(a^2-b^2)^2n^2=0.$ (*Vide* Askwith, p. 182, Ex. 33, where the point equation is given.)

If (x_1, y_1) is the orthocentre of $\Delta\ QRS$, polar circle of $\Delta\ QRS$ is, say, $S' \equiv x^2+y^2-2xx_1-2yy_1+x_1^2+y_1^2-r^2=0.$

Now, S is harmonically circumscribed to S'. So their mutual θ invariant is zero, that is

$$(x_1^2-r^2)\cdot\frac{1}{a^2}+(y_1^2-r^2)\cdot\frac{1}{b^2}-1=0. \tag{1}$$

Also, Σ is harmonically inscribed to S'. Hence

$$a^6+b^6=(a^2-b^2)(x_1^2+y_1^2-r^2)=0. \tag{2}$$

Eliminating r from (1) and (2) we have

$$\frac{a^6+b^6}{(a^2-b^2)^2}=x_1^2+y_1^2-(b^2x_1^2+a^2y_1^2-a^2b^2)/(a^2+b^2)$$
$$=a^2x_1^2+b^2y_1^2+a^2b^2.$$

Hence the required locus is

$$a^2x^2+b^2y^2=\frac{a^6+b^6}{(a^2-b^2)^2}-a^2b^2$$
$$=\left(\frac{a^4+b^4}{a^2-b^2}\right)^2.$$

Trichy, S. VENKATASUBRAMANIAN,

REVIEWS

Modern Operational Calculus with Applications in Technical Mathematics

BY N. W. MCLACHLAN

(Pp. xiv + 218, 1948, Macmillan & Co., London)

This book is intended to popularise the methods of operational calculus and apply them to Technical Mathematics. Modern Engineers and Technologists cannot afford to be content with the thumb-rules dictated by professional mathematicians and must themselves possess and cultivate a broad knowledge of mathematics with reasonable vigour so that they may be in a position to apply mathematical methods and formulæ correctly and appropriately in their practical work. The operational method introduced by Heaviside with a view to simplify mathematics for the practical physicist in electromagnetic problems has now been made logical and unambiguous by the introduction of the Laplace Transform, abridged as *L.T.* The *L.T.* of any function $f(t)$ is defined by the integral $p\int_0^\infty e^{-pt}f(t)\,dt$, or $\phi(p)$, say, which is said to be *interpreted* as $f(t)$. A theorem of Lerch comes in handy to show the uniqueness of this transform under certain conditions and a few simple rules enable us to pass from the *L.T.* of $f(t)$ to the L.T.'s of associated functions like $e^{at}f(t)$, $\frac{d^n}{dt^n}f(t)$ and $\int_0^\infty f(t)\,dt$. The L.T.'s transform ordinary differential equations into algebraic equations and partial differential equations with initial and boundary conditions into ordinary differential equations, thus reducing calculus to algebra in successive stages. The examples provided in this book show how an electrical engineer can solve easily by the L.T. method the differential equation for the current in a circuit or for transmission of electrical energy in a uniform sub-marine telegraph cable, while a radio-engineer can attack by the same method the differential equation for sinusoidal sound waves of very small amplitude.

All the above matter and more are comprised in seven chapters of about seven-score pages. The first two chapters contain preliminary matter explaining the nature of the Laplace Transform and manipulations with the Laplace Integral. The third chapter deals with simple and simultaneous ordinary linear differential equations of order n with constant coefficients and furnishes practical applications from non-dissipative oscillatory electrical circuits and simple electrical networks. The fourth chapter gives the solution of simple and simultaneous partial linear differential equations with constant coefficients, which occur in a number of technical applications. Validity of solutions obtained is proved in each case. There are also a sufficient

number of technical examples to illustrate practical applications. Chapters V and VI are concerned with the evaluation of some difficult integrals, the establishment of mathematical relationships such as the expansion of $J_v(u)$ and $\cos\left(t - \frac{\pi}{4}\right)$ in terms of Laguerre polynomials, and the derivation of Laplace Transforms by devices like Mellin Inversion and functional differential equations. The last and the seventh chapter deals with periodic impulses of finite duration and Fourier expansions with practical examples from dynamics and electricity. The problems at the end constitute the eighth chapter though not explicitly mentioned as such, and include important formulæ and additional theorems. The reader who knows Bessel functions can easily solve a large proportion of these problems.

The book concludes with a list of 81 important Laplace Transforms, 17 references and a useful index.

Two special features of the book are the abundant use of graphs to illustrate all the peculiarities of functions and a pure mathematical Appendix intended to supply 'mathematical limit gauges,' so to speak, to check the validity of analytical operations.

The typography is excellent and the presentation is a model of clarity and simplicity. The book is indispensable to students of advanced engineering and technology.

A. A. K.

Co-ordinate Geometry

BY S. C. CHAKRABARTI

(Pp. 466, Price Rs. 7-8-0. H. Chatterjee & Co., Calcutta)

The subject-matter comprises the study of the conic sections by using Cartesian and polar co-ordinates, so as to serve the requirements of the B.A. and B.Sc. students of Indian Universities. The brief chapters on three-dimensional analytical geometry, including the plane, the straight line and the second degree surfaces of revolution will be very useful to students of several universities.

It would be presumptuousness on the part of any one to claim or certify originality within the realm of this familiar and well-thrashed-out subject, but it is no mean compliment if it is said that this book is not a copy of any of the numerous books on this subject. It is intended as a text-book, but the mediocre student and the mediocre teacher who expects the subject-matter just necessary and sufficient for examination purposes, to be presented in neatly arranged spoonfuls to be swallowed in the order given, may not find this a text-book to his liking. But the intelligent students and teachers

who are able to read the book backwards and forwards will find the book extremely interesting and illuminating. Simpler and more elegant methods can doubtless be pointed out in many places, and a different arrangement of the subject-matter would make it more appealing. These should not, however, detract from the value of a really refreshing book. A large number of interesting examples (with hints), not usually found in the common text-books, forms a feature of this book.

One will not feel happy, if he is told that in a book of 466 pages, 145 pages are devoted to the straight line. This portion could have been condensed, but the later portions of the subject have received adequate attention, and within the limits of the syllabus covered by the author, no topic has escaped treatment.

Under the present facilities for printing mathematical books in India, one has to put up with the large number of printing mistakes. The author, instead of remarking "All minor errors being obvious are left to the reader for correction" should have held out his patience a little more, and compiled a list of errata.

"TEACHER."

Higher Certificate and Intermediate Tests in Pure Mathematics

BY R. J. FULFORD

(Pp. 127, Second Edition, 1948, University Tutorial Press, London)

This is an excellent collection of 516 examples in Pure Mathematics ranging from the Intermediate to the B.A. and the B.Sc. pass standards and covering topics in Algebra, Trigonometry, Analytical Geometry, Calculus, Plane Geometry, and Solid Geometry. The book is divided into 4 parts of which Part I contains easy questions. Parts II and III contain straightforward questions of a somewhat higher standard, while the more difficult questions 120 in number go into Part IV. While Parts I and II are arranged according to subjects, Part III which is called "General" and Part IV are arranged into collections each of 10 examples covering topics in all subjects. Answers are provided at the end of the book, but there are no hints for solution. The get-up is attractive and the booklet should be of great help to teachers in selecting questions whether for the classroom or for examinations.

A. N. RAO.

31/-49 Printed at The Bangalore Press, Bangalore City, by G. Srinivasa Rao, Superintendent.

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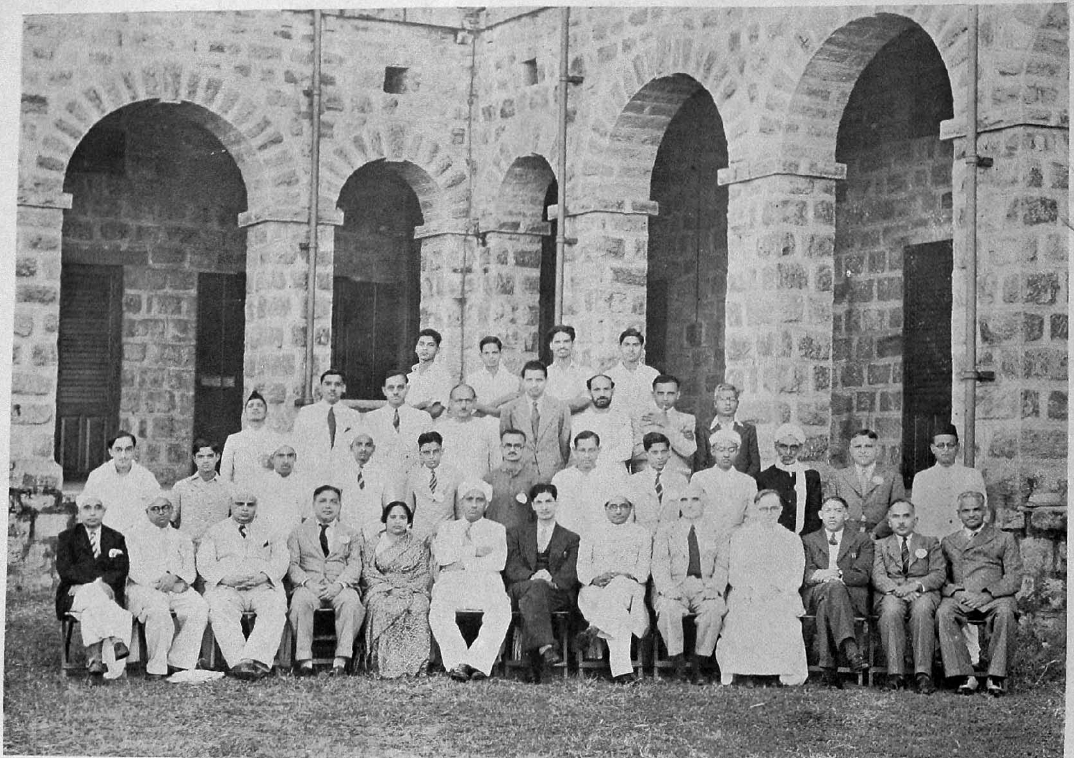
ERRATA

For Volume XV

Page	Line	For	Read
20	5	$\int_{N-1}^{n} (x)\, dx$	$\int_{N-1}^{n} f(x)\, dx$
21	14	$f(\zeta)$	$f(\xi)$
,,	20	$\sum_{k=1}^{\infty}$	$\sum_{r=1}^{\infty}$
22	11	$\sim$	$=$
23	8	n'_2	n_2
,,	14	$n_r - k_r$	$n_r - 1$
,,	16		insert bracket }
25	16	$=$	$\leqslant$ (in all four places)
40	22	$3n$	3^n
,,	30	x	x^2
43	Footnotes	After Landau	*Ergebnisse der Funktionentheorie* (1916), p. 61.
44	3	$3n$	3^n
45	8	$>$	$<$
46	16	Third	Fourth
,,	,,	141	157
47	20	d/dr	D^r
48	7	x	h
,,	8	after " it is "	read " also independent of x. It is then equal to $\frac{1}{n+1}$ or in the alternative has any value."
50	15	xr	x_r
77	38	with the help of J. L. Coolidge.	with industry almost superhuman.
92	15	$\cos \theta$	$\cos \theta/2$
,,	,,	$\sin \theta$	$\sin \theta/2$



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